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Reverse transient of a p-n junction

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REVERSE TRANSIENT OF A P-N JUNCTION

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I. INTRODUCTION

In this complex age there is a demand for handling enormous amounts of information in very short time periods. To meet this need, those in the computer industry are constantly searching for techniques to increase the already tremendous data processing speed of their product. Considering the complexity of the overall computer, there are obviously many factors contributing to its operating speed. However, one of the most basic factors is that of the decision speed of the individual components. It is with this consideration that this thesis will be concerned.

Before the computer can make a decision, many individual diodes and transistors must be switched from one electrical state to another. Since no implemented device is a theoretical switch, there is an inherent delay associated with each element. In this thesis, the switching delays will be analyzed and an investigation made of how these delays vary as a function of the specified parameters of the diode. These results may be used by the circuit designer and the component engineer to predict switching delays for a specified p-n junction.

A. Explanation of the Switching Transient

To analyze the switching transient, both the forward and reverse bias conditions must be considered. The forward conductivity of the diode is usually large enough and the forward junction voltage is usually small enough that the forward current is determined by the external circuitry. Under this consideration the delay associated with forward biasing of a diode may be neglected. Even if these conditions are not met, the delay of the junction current to reach its steady state value for forward bias is negligible small compared to that of reverse bias. However, it is still necessary to study the forward bias condition. This is because the conditions at the end of the forward bias must be found and applied as the initial conditions for the reverse bias state.

An equivalent circuit of the switching process, together with applied voltage and diode current waveshape, is shown in Figure 1. Before the time $T = 0$, the diode may be considered either open circuited or in the steady state reverse bias condition.

At time $T = 0$ a forward bias current pulse of I_F is applied to the diode. During this period, positive current carrier (holes) are passed through the p-type material and injected into the n-type material at the junction. This results in an excess of holes in the n-type region, the concentration of which is a maximum at the junction and decays to zero far away from the junction. The carrier concentrations during this forward bias condition are shown in Figure 2.

At time $T = T_F$, the forward bias is terminated and a reverse voltage V is applied to the circuit. For a finite period after the application of the reverse bias, there remains a concentration of excess holes in the vicinity of the junction. These holes serve as current carriers which prevents the junction voltage from reversing. This effect causes the diode to behave as a short. Hence the value of the junction current is a constant; equal to the applied reverse voltage divided by the external circuit resistance. This period will be called the storage time of the reverse transient.

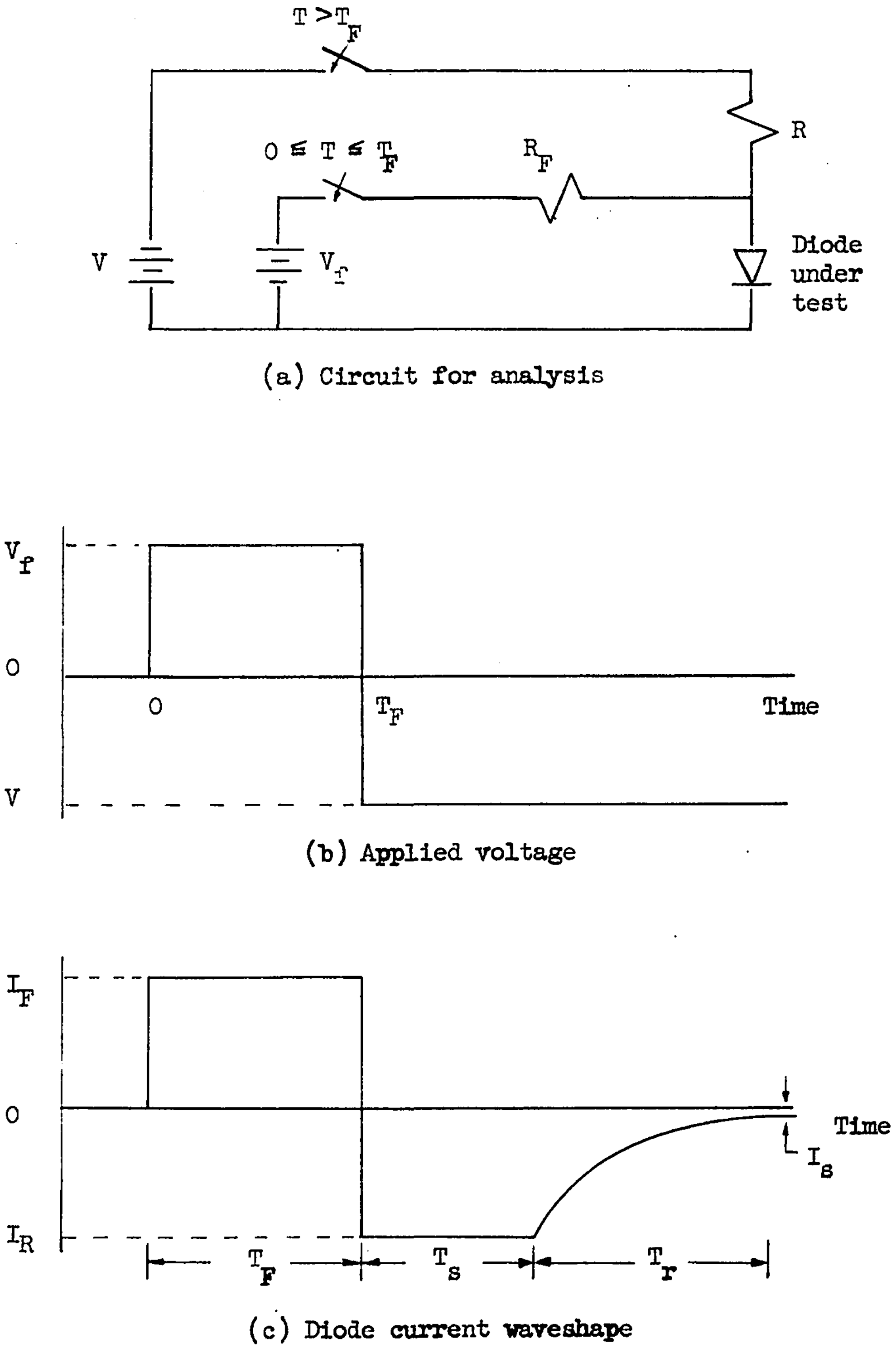


Figure 1. Circuit and waveshapes for the switching transient

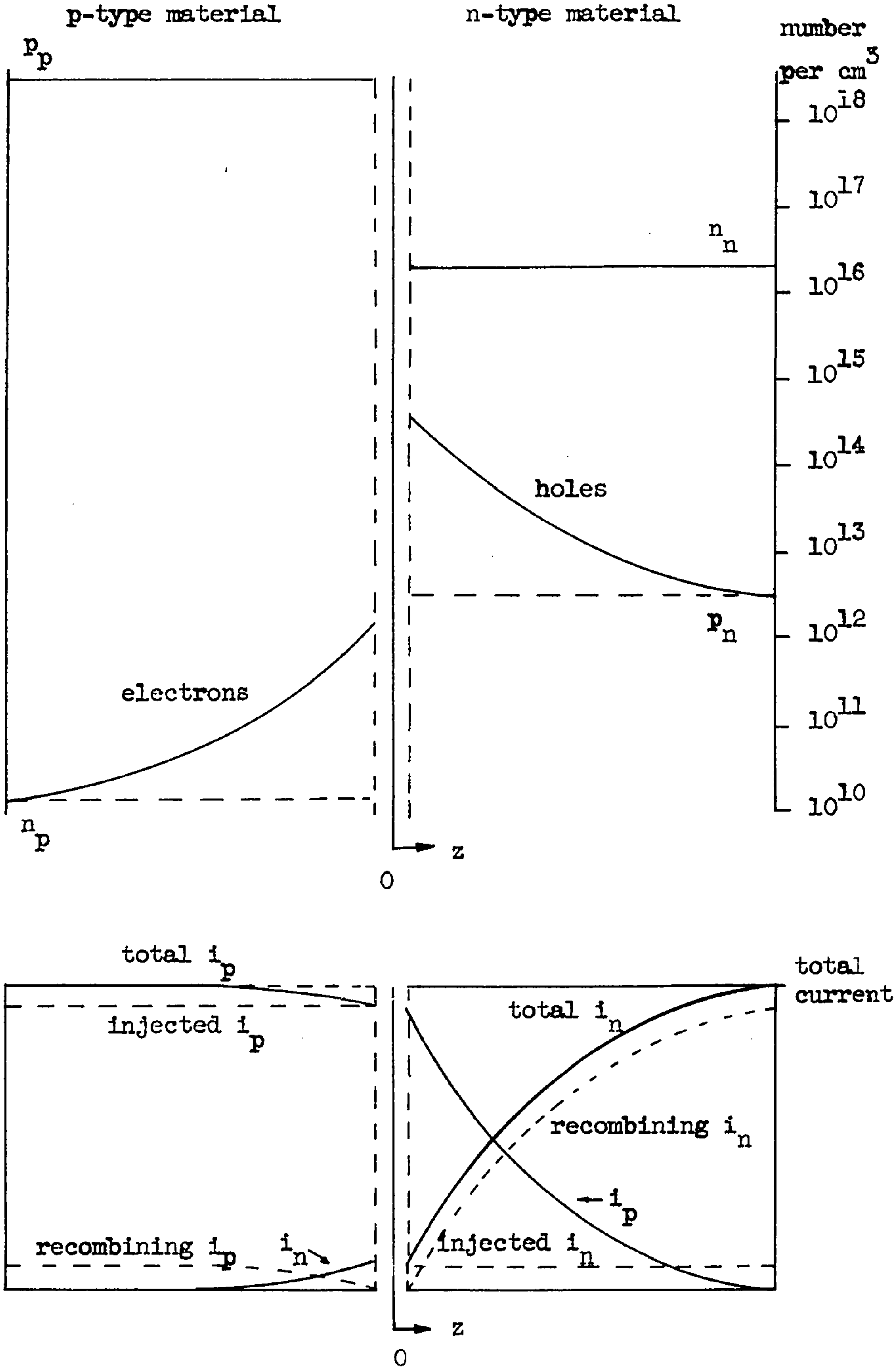


Figure 2. Carrier concentration and diode current during forward bias (10)

After a time of (T_S), the carrier density at the junction falls to the equilibrium value. The junction voltage then reverses and begins to rise toward the value of the applied reverse voltage. The time after the storage period is defined as the recovery phase. The current during this portion results from removing the remaining excess holes in the n-type region and charging the depletion layer capacitance. A sketch of the junction voltage and diode current is shown in Figure 3.

B. Definition of Terms and Symbols

In the calculations, it will be advantageous to define the value of time as zero at the beginning of each phase. However, a distinction must then be made between the various phases of operation. For this purpose the period of forward bias will be denoted as Phase I, the storage time as Phase II and the recovery period as Phase III. Each quantity in this thesis with a subscript of I, II, or III will mean that it is defined for its corresponding phase with $T = 0$ being taken as the beginning of that phase.

To reduce the number of terms appearing in the equations, normalized time and distance will be used throughout this analysis. The normalized time (T) will be defined as the actual time (t) divided by the average lifetime of the holes in the n-type material. The normalized distance (z) will be defined as the actual distance through the material (x) divided by the average diffusion length (L_p) of holes in the n-type material.

For ease of handling in the differential equations, the symbol (I) will be defined as a dimensionless quantity proportional to current. Although it will be called current, the value of (I) shown in this thesis

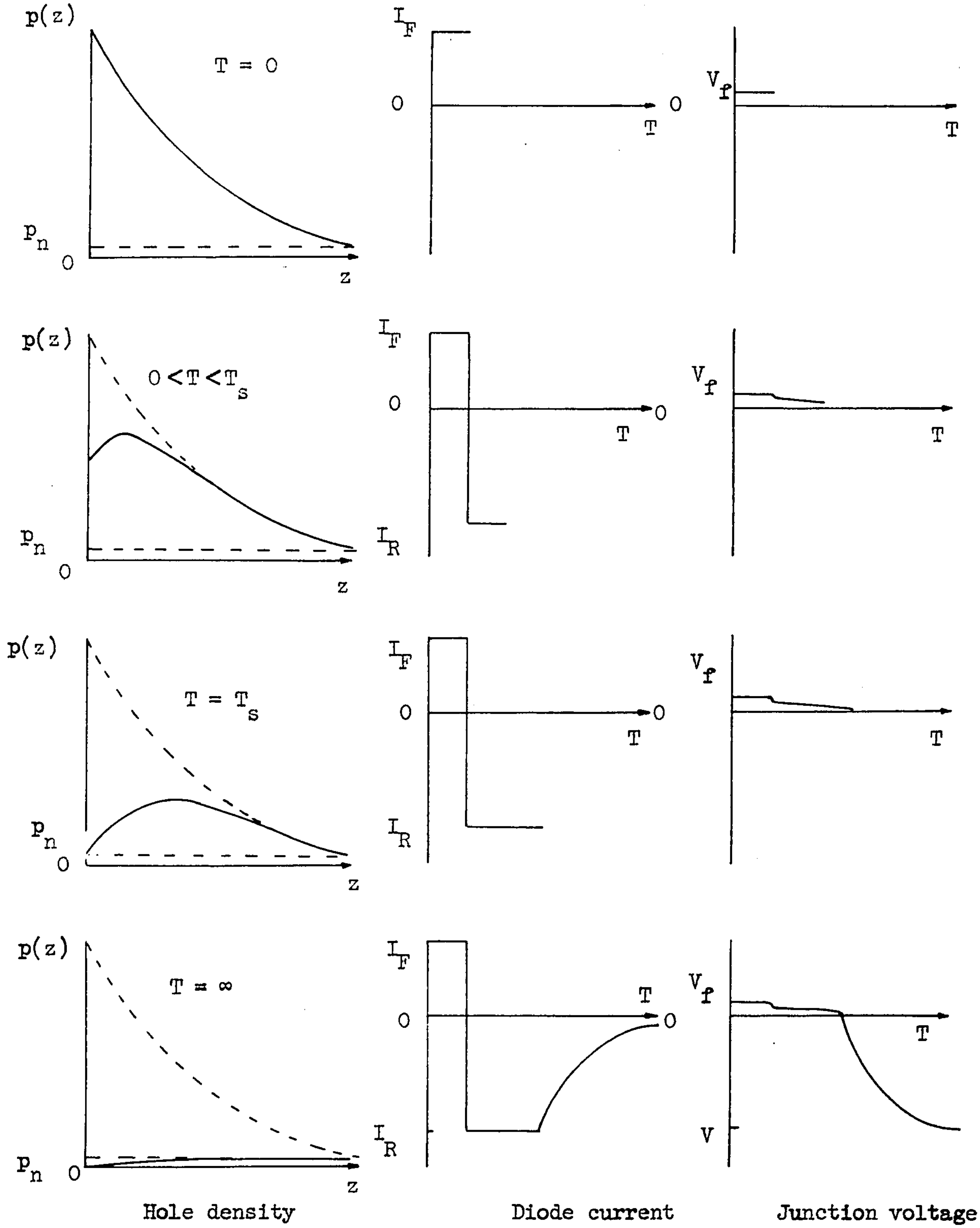


Figure 3. Hole density, diode current, and junction voltage during reverse bias

must be multiplied by qD_p/L_p to obtain the actual current density (amps/m^2).

A complete list of defined terms and symbols may be found in Table 1.

Table 1. Definition of symbols

$I = -\frac{\partial p}{\partial z}$ - This is defined as a dimensionless quantity proportional to current. ($J_p = -qD_p\, \partial p/\partial x$: $I = J_p L_p/qD_p = -\partial p/\partial z$)

I_F - Forward current flowing before application of reverse bias.

I_R - Initial reverse current flowing through the diode after the reverse bias is applied. This current is determined by the external circuitry and flows during the constant current phase of recovery. This current is defined as negative.

θ - Ratio of the reverse current immediately after the application of reverse bias ($-I_R$) to the forward current flowing immediately before the application of the reverse bias (I_F).

v_J - Junction voltage of the diode.

V - Reverse bias voltage applied to the diode circuit.

T - Normalized time (t/T_p).

T_s - Normalized storage time of the junction. Time from the application of the reverse bias until the junction voltage goes through zero negatively.

T_p - Average lifetime of the holes in the n-type region.

D_p - Diffusion constant for holes in the n-type region.

u_p - Mobility of holes in the n-type region.

L_p - Diffusion length of holes in the n-type region. ($L_p = D_p T_p$)

x - Distance through the one-dimensional semiconductor. ($x = 0$ at the junction)

z - Normalized distance through the crystal. ($a = x/L_p$)

p(T,z) - Hole density (in number of holes per cm^3) as a function of the normalized distance through the crystal and time.

Table 1. Definition of symbols (continued)

p_n'	- Total density of holes in the n-type region.
p_n	- Equilibrium hole density in the n-type region.
p	- Excess hole density in the region of consideration. ($p = p_n' - p_n$)
S	- Recombination velocity (cm/sec)

II. PREVIOUS ANALYSIS OF THE SWITCHING TRANSIENT

A major contribution to the understanding of the response of a p-n junction resulted from work conducted concurrently by R. H. Kingston and by B. Lax and S. F. Neustadter (6, 8) at the Lincoln Labs in 1953-54. In their articles, the storage time of a p-n junction was found to be related to the values of forward and reverse current by the error function relation of erf $\sqrt{T_s} = 1/(1 + \theta)$. This analysis, however, was only conducted for the simplest mathematical model with the following properties: 1. Infinite length of n-type region; 2. No potential gradient in the n-type region; 3. Steady state forward bias.

In solving for the current during the recovery phase severe assumptions were made to obtain a closed solution. Also this solution had a singularity at the end of the storage phase and hence was only good for large values of time beyond the storage period. To compensate for this, an expression was obtained for the diode current if an infinite amount of reverse current was initially allowed to flow, i.e., $T_s = 0$. The conclusion was that the current during the recovery phase would always be less than predicted by the solution for this period and greater than the equation for the diode current following a zero storage time period. The use of this limiting case of $T_s = 0$ was also employed by Shulman and McMahon (13).

The investigation of the reverse transient after a finite forward bias time was conducted by W. H. Ko (7). However, after the equations were set up to determine the storage time the following statement was made: "Because of the complexity of the initial condition as well as the implicit relationship between the boundary conditions, it has not been found possible to obtain a simple exact solution for this problem". Hence, approximations were made which limited the application of the results.

All of the previously mentioned treatments of the switching transients have assumed constant minority carrier lifetime (Tp). To investigate this assumption, methods have been derived to measure this quantity (9, 14). However, the assumption of a constant value for the hole lifetime is not always true. The variation in minority carrier lifetime as a function of the hole energy level has also been investigated, (1, 2 and 12).

III. SCOPE OF INVESTIGATION

With the variations in physical dimensions of the diode material and the variations in operating conditions, many diode applications do not fall within the assumptions made for the existing treatments. By considering these factors, it is the purpose of this thesis to serve as a general reference for predicting the reverse response of the p-n junction.

A. Assumptions

Some of the general assumptions made in the cited literature are the following:

1. Steady state forward bias condition prior to the application of the reverse bias pulse.
2. The width of the n-type material is much greater than the diffusion length of minority carriers in that region.
3. No field intensity away from the junction.
4. During the recovery phase, the current associated with the creation of the depletion layer capacitance can be neglected compared to the diffusion current through the junction.
5. The lifetime of the holes in the n-type material is a constant throughout the switching transient.
6. The conductivity, consequently the doping level, of the p-type material is much greater than that of the n-type material.

In this analysis, a complete investigation will be conducted for the condition when assumptions 1, 2, and 3 are not met. In other words, the case will be considered when the forward bias pulse is applied for a finite period T_F, the width of the n-type region (W_n) is comparable to the diffusion length of the minority carriers and when there exists a potential gradient in the n-type region.

The validity of the assumption 4 will be investigated. Conclusions will be drawn as to when this assumption may be employed and what compensations must be made when the approximation is not warranted.

Assumptions 5 and 6 above will be used here as they have been in all literature cited. Assumption 6 is invariably met in commercially available diodes and does not limit the application of the results. A complete analysis of the validity of assumption 5 is given by Shockley (12).

B. Method of Analysis

A general review of transport phenomena will first be conducted. From this consideration the diffusion equation, which governs the flow of positive carriers in the n-type region, will be obtained.

The solution of the diffusion equation yields terms which contain the error integral or error function (as it is usually called). This function appears several times in this thesis in the boundary conditions of subsequent differential equations. Due to the difficulty in handling the error function in differential equations it became necessary to obtain an approximation for it. By making an exponential approximation, exact solutions to the necessary differential equations may be obtained.

From the use of the diffusion equation, the error function approximation and the proper boundary conditions, the reverse response of the diode will be determined.

IV. ANALYSIS AND CALCULATIONS

A. General Transport Theory

The basic equation governing the flow of particles is the equation of continuity. This equation, which results from the conservation of matter, may be stated as follows: The rate of increase of particles within a volume is equal to the net inward flow across the surface of the volume. The continuity equation applied to the minority carrier holes in the n-type region may be stated as follows:

$$\left|\begin{array}{l}\text{time rate}\\ \text{of increase}\\ \text{of holes}\end{array}\right| = \left|\begin{array}{l}\text{rate of ther-}\\ \text{mal generation}\\ \text{of holes}\end{array}\right| - \left|\begin{array}{l}\text{rate of re-}\\ \text{combination}\\ \text{of holes}\end{array}\right| - \left|\begin{array}{l}\text{divergence}\\ \text{of hole}\\ \text{flow}\end{array}\right| \qquad (1)$$

The method of the derivation presented here is similar to that of references 10, 11 and 15. We will consider a small volume, in the n-type material of dx dy dz and centered at x, y, z. The rate of change of holes in dx dy dz is

$$\frac{\partial p}{\partial t}\, dx\, dy\, dz \qquad (2)$$

which becomes the term on the left hand side of Equation 1. The excess rate of generation over recombination may be written as

$$(g - r)\, dx\, dy\, dz\ ; \qquad (3)$$

where g is the net rate of generation of holes per unit volume (due to thermal excitation, etc.) and r is the net rate of recombination with electrons. This term represents the first two quantities on the right hand side of Equation 1.

The net flow of holes into the region may be obtained from the current density. Since by definition the region extends in the x direction from $x - dx/2$ to $x + dx/2$, the current density into the dy dz face of the volume will be

$$I_{px}(x, y, z) - \frac{\partial I_{px}}{\partial x}(x, y, z)\frac{dx}{2} \tag{4}$$

and that out of the other will be

$$I_{px}(x, y, z) + \frac{\partial I_{px}}{\partial x}(x, y, z)\frac{dx}{2} \quad . \tag{5}$$

The resulting hole flow into the region will then become

$$\frac{1}{q}\left(I_{px} - \frac{\partial I_{px}}{\partial x}\frac{dx}{2}\right) dy\, dz - \frac{1}{q}\left(I_{px} + \frac{\partial I_{px}}{\partial x}\frac{dx}{2}\right) dy\, dz$$

$$= -\frac{1}{q}\frac{\partial I_{px}}{\partial x}\, dx\, dy\, dz. \tag{6}$$

Similarly the net hole flow into the dx dy and dy dz faces may be found, which leads to the net flow into dx dy dz being

$$\frac{1}{q}\left(\frac{\partial I_{px}}{\partial x} + \frac{\partial I_{py}}{\partial y} + \frac{\partial I_{pz}}{\partial z}\right) dx\, dy\, dz = \frac{1}{q}\nabla \cdot \vec{I}_p\, dx\, dy\, dz \ . \tag{7}$$

Equation 7 now becomes the last term on the right hand side of Equation 1. Substituting these values into the continuity equation, one obtains

$$\frac{\partial p}{\partial t} = (g - r) - \frac{1}{q}\nabla \cdot \vec{I}_p \quad . \tag{8}$$

The following quantities will now be defined:

$$\begin{aligned} p_n &= \text{equilibrium hole density in n-type region} \\ p' &= \text{total hole density in the n-type region} \\ p = p' - p_n &= \text{excess holes density} \end{aligned} \tag{9}$$

Also, the assumption will be made that the number of excess holes will decay with the characteristic lifetime (T_p) which is independent of the concentration. The following may now be written:

$$g - r = \frac{p_n - p'}{T_p} . \tag{10}$$

The continuity equation now becomes:

$$\frac{\partial p}{\partial t} = - \frac{p}{T_p} - \frac{1}{q} \nabla \cdot \vec{I}_p + g_p' , \tag{11}$$

where g_p' represents the net rate of hole generation due to external effects such as photons, etc. In this thesis, this term will be neglected.

If a semiconductor region is under the influence of an electric field, the hole current is given by (1):

$$\begin{aligned} \vec{I}_p &= \text{drift current} + \text{diffusion current} \\ &= q \mu_p \vec{E} (p + p_n) - q D_p \nabla p . \end{aligned} \tag{12}$$

Substituting this into Equation 11,

$$\frac{\partial p}{\partial t} = - \frac{p}{T_p} - \mu_p \nabla \cdot \vec{E} (p + p_n) + D_p \nabla^2 p . \tag{13}$$

For the investigation here, only current flow in the x direction will be considered and thus ∇p may be replaced by $\frac{\partial p}{\partial x}$. By using the following relationships

$$z = \frac{x}{L_p} = \frac{x}{D_p T_p} \qquad \text{(Defined)},$$

$$T = \frac{t}{T_p} \qquad \text{(Defined)} , \tag{14}$$

and

$$\mu_p = \frac{qD_p}{kT^o} \qquad \text{(Einstein's Relationship).}$$

Equation 12 becomes

$$\frac{\partial p}{\partial T} = \frac{\partial^2 p}{\partial z^2} - f\, L_p \frac{\partial p}{\partial z} - p. \quad \text{Where } f = \frac{q\, E}{kT^o} \;. \tag{15}$$

This equation, which is commonly known as the diffusion equation, will be used throughout this thesis to determine the concentration of excess holes during the reverse transient.

B. Error Function Approximation

When a current is specified as a boundary condition of the diffusion equation, the solution will always contain error function terms. Since the diffusion equation predicts the current flow during the entire switching period, one might conclude that the error function will be in evidence in the results throughout this thesis. If no attempt is made to replace it with a more easily handled expression, the results will appear in the form of integrals or infinite series and will be of limited practical use. With this in mind, an approximation to the error function will be obtained. To prevent this approximation from seriously limiting the accuracy of the results, the procedure will be made available such that any degree of accuracy desired may be obtained.

A polynomial approximation for the error function is given by Hildebrand (4). However, this series is very slow to converge for certain values of the argument. The convergence may be improved by defining a different polynomial for various ranges of the argument. However, when an entirely different series is used for exclusive range, it becomes awkward to apply to a general analysis where the argument may take on all values.

A more convenient form for the approximation is the exponential. This is because the exponential, or a series of them, is easily ahndled in the differential equations encountered in this analysis. This form is also suggested by the fact that $(1 - e^{-x})$ is a rough approximation commonly used for the error function of x. The procedure to obtain an approximation of the following form will now be considered:

$$\text{erf } x \approx \sum_{i=1}^{N} C_i \, e^{a_i x} \text{ for } 0 \leq x \ . \tag{16}$$

Intuitively one can reason that all the a's will be negative or zero and the C's finite. However, if this is to be a general representation negative values of x must be considered. Since the a's are negative, the sequence shown in Equation 16 will not be bounded for negative x.

For the consideration in this problem, the defining equation for the error function and the error function identities shown below will be used.

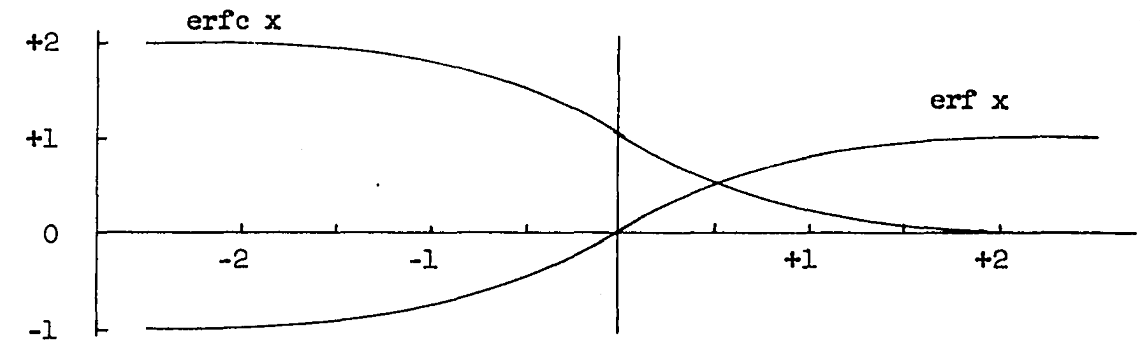
$$\text{Error function of } x = \text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} \, dy \ . \tag{17}$$

$$\text{Complementary error function of } x = \text{erfc } x = 1 - \text{erf } x. \tag{18}$$

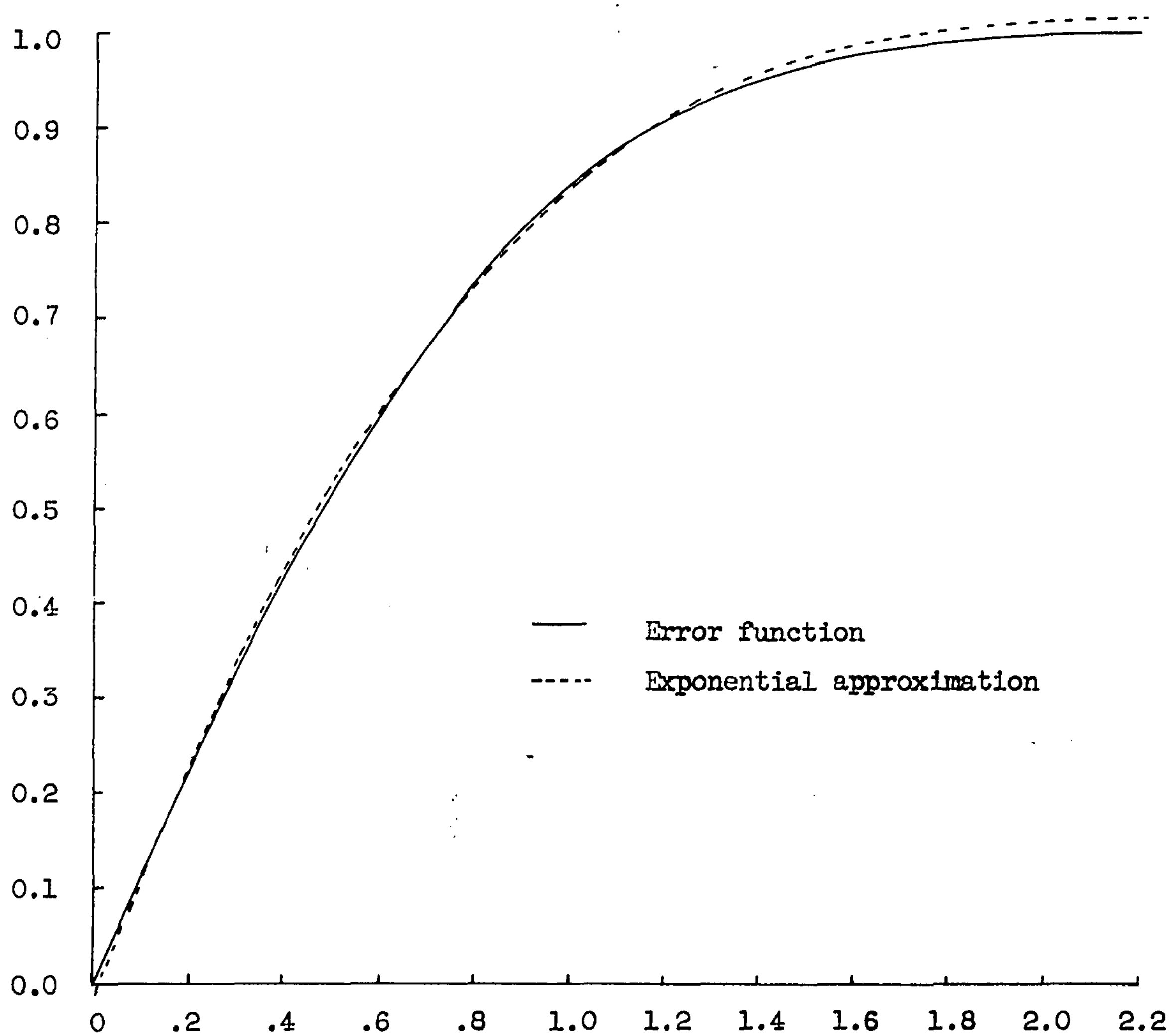
$$\text{erf } (-x) = - \text{erf } x \ . \tag{19}$$

$$\text{erfc } (-x) = 1 + \text{erf } x \ . \tag{20}$$

These equations are illustrated in Figure 4. It now may be seen that the difficulty with the negative values of the argument may be overcome in one of the following ways.



(a) Error function and complementary error function



(b) Error function and approximation

Figure 4. Error function with exponential approximation

1. In solving for the C's and a's in Equation 16 use values of erf (x) for both positive and negative values of x. In other words, match the curve from large negative values to large positive of x.

2. Using the above identities, rewrite Equation 16 to take into account the sign change as follows:

$$\text{erf } x = \sum_{i=1}^{N} \frac{x}{|x|} C_i e^{a_i |x|} . \tag{21}$$

3. Carry out the calculations assuming the argument is positive. Then, with the use of the above identities, note the sign changes in the results that would occur if the argument were negative.

In consideration of the above possible approaches one can conclude that the first would lead to matching the approximation to the error function at a great number of points. This would lead to such an enormity of calculations that it must be ruled out.

Although the sequence shown in number 2 will have the proper sign change to represent the error function for all values of the argument, it would be very awkward to handle in differential equations. If this representative were used, it would be necessary to state whether the argument were positive or negative before the differentiation could be performed. Because of this, the second approach has no advantage over the third.

In this thesis, the third approach will be used. The exponential approximation will be calculated for positive arguments of the error function.

When this approximation is used in the differential equation, the solutions will be obtained assuming the argument is positive. Once the results have been found the sign changes will be noted for the case when the argument of the error function is negative. The exponential approximation may then be written as:

$$\text{erf } x = \sum_{i=1}^{N} C_i \, e^{a_i x} \quad \text{for } 0 \leq x \, , \tag{16}$$

and

$$\text{erf } x = \sum_{i=1}^{N} - C_i \, e^{-a_i x} \quad \text{for } 0 > x \, . \tag{22}$$

In obtaining this approximation, both the C's and a's in Equation 16 will be assumed to be undetermined. One approach in finding these values is to minimize the square of the difference between the approximation and the error function as the value of each C and a is varied. By using computer techniques, these calculations may be carried out until the approximation is within the desired accuracy of the error function over the specified range. Because of the non-repeatability of the solution and the extensive calculation which must be undertaken to improve the accuracy of the approximation, this approach will not be used here.

Instead, a sequence of N exponentials will be set equal to the error function at 2N equally spaced points. Two N points are needed since each exponential has two unknowns; the coefficient of the term (C) and the coefficient of the power (a). This procedure leads to exact solutions for the C's and a's with the approximation being equal to the error function at the selected points. This procedure, however, has the shortcoming that the a's may take on all values (real or imaginary, positive or negative). But, for application in this analysis it is necessary for each a to be negative and real. Hence for use here, the value of the a's which are nearest to the value dictated by the equations and yet are real and negative will be selected.

The accuracy of the approximation may then be checked for all values of the argument. If the accuracy calculated is not great enough, more terms may be added to the approximation with two equating points added for each new term. This procedure yields itself quite easily to increasing the accuracy to any desired value by increasing the number of terms in the approximation. This is especially true if a computer is available with subroutines for sulution to N equations with N unknowns.

A complete outline of this procedure is shown in Appendix A. In this appendix a 4 term exponential series is calculated as an approximation to the error function. This approximation is tabulated in Table 2 and plotted in Figure 4.

C. Forward Bias

At the beginning of the forward bias period, a step of current I_F is forced through the diode in the forward direction. This assumption does not limit the application of this material for most industrial use. The reason being that the equivalent external resistance (R_F in Figure 1) is generally much greater than the forward bias resistance of the diode.

With this assumption, positive carriers will be injected into the n-type material at a constant rate. The motion of these carriers is governed

Table 2. Error function approximation

$$1 + 3.9262e^{-1.7835x} - 10.8287e^{-1.3725x} + 5.8804e^{-1.1160x} \approx \text{erf } x$$

x	Approximation	erf x	Difference
0.0	-.02210	.00000	-.02210
0.1	.10435	.11246	-.00811
0.2	.22303	.22270	+.00033
0.3	.33275	.32863	+.00412
0.4	.43290	.42839	+.00451
0.5	.52333	.52050	+.00283
0.6	.60420	.60386	+.00034
0.7	.67587	.67780	-.00193
0.8	.73887	.74210	-.00323
0.9	.79381	.79691	-.00310
1.0	.84136	.84270	-.00134
1.1	.88217	.88021	+.00196
1.2	.91693	.91031	+.00662
1.3	.94627	.93401	+.01226
1.4	.97082	.95229	+.01853
1.5	.99113	.96611	+.02502
1.6	1.00775	.97635	+.03140
1.7	1.02114	.98379	+.03735
1.8	1.03176	.98909	+.04267
1.9	1.04000	.99279	+.04721
2.0	1.04619	.99532	+.05087
2.1	1.05067	.99702	+.05365
2.2	1.05370	.99814	+.05556
2.3	1.05552	.99886	+.05666
2.4	1.05625	.9990	+.05725
3.0	1.04862	.99998	+.04864
5.0	1.01102	.99999	+.01103

by the diffusion equation which was derived previously as Equation 15. Most treatments of the solution of this equation consider only the steady state forward bias case, which permits letting $dp/dT = 0$. In this treatment the time dependent solution will be found and the time of forward bias will be specified as T_F. Considerations will then be made for the case of large T_F (steady state) and also for small T_F (transient forward bias condition).

For this forward bias case, the hole density satisfies the following equation subject to the shown boundary conditions:

$$\frac{\partial p_I}{\partial T} = \frac{\partial^2 p_I}{\partial z^2} - p_I \tag{23}$$

$$p_I(0, z) = 0 \tag{24}$$

$$- \left. \frac{\partial p_I}{\partial z} \right|_{z = 0} = I_F \tag{25}$$

and

$$p_I\,(T, \infty) = 0. \tag{26}$$

Equation 23 is the diffusion equation with the modification that f = 0. This term is dropped here since presently the case of no field intensity away from the junction is being considered. The case of a built in field will be considered later.

The first boundary condition (Equation 24) states that at the time of application of the forward bias, there are no excess minority carriers in the n-type region. The second (Equation 25) defines the externally determined forward current. The last boundary condition states that the width of the n-type material is great enough that the excess hole concentration is zero at the n-type ohmic contact.

To aid in solving the above equation, Laplace Transforms will be used. The time dependance of the equation will be transformed and the differential equation in z solved. This solution will be a function of z and s (the transformed variable). The solution to the original problem then may be obtained by performing the Inverse Laplace Transform.

The calculation for the solution to the forward bias period is carried out in Appendix B. The following equation is the result:

$$p_I(T,\ z) = \frac{I_F}{2}\left[e^{-z}\ \mathrm{erfc}\ \left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z}\ \mathrm{erfc}\ \left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right]\ . \qquad (27)$$

In accordance with the prescribed subscripts, this will be denoted as Phase I. The values of T will be the normalized time of forward bias with T = 0 being the time of application of the forward bias.

From Equation 27, a plot can be made of the dimensionless quantity p/I_F as a function of z. This plot is shown in Figure 5 with the values for the time of forward bias being 0.0, 0.05, 0.1, 0.2, 0.5, 1.0, 2.0, 5.0, and infinity. From this plot it may be seen that the slope of the hole concentration is a constant at the junction for all values of time. This, of course, is due to the diode current during this period being a constant.

For large values of T (greater than 3), the hole concentration shape assumes a decaying exponential. This also may be seen by letting the value of T approach infinity in Equation 27. This would correspond to the steady state forward bias condition. From the error function identities shown previously, the following is the result.

$$p(z) = I_F\ e^{-z}\ . \qquad (28)$$

The correspondence between this equation and that given by other authors may be seen by noting that our I_F (defined in Table 1) is L_p/qD_p times the actual current density. Since this current density is given by Middlebrook (10) as

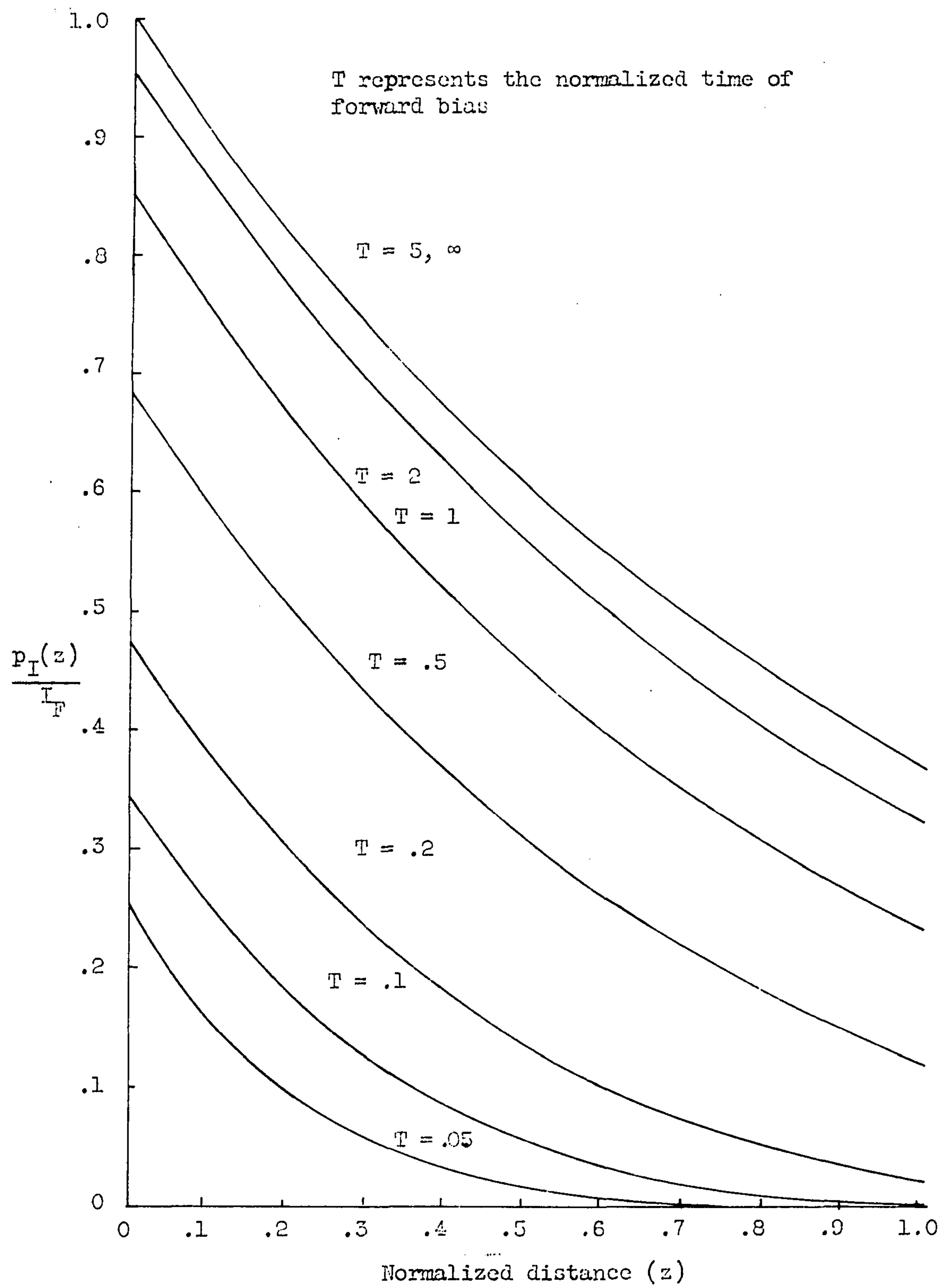


Figure 5. Excess hole density during forward bias

$$J_p = \frac{qD_p}{L_p} \; p_n \, (e^{\frac{qv_J}{kT^o}} - 1) \; . \tag{29}$$

Equation 28 may be written in the more familiar form of

$$p = p_n \, (e^{\frac{qv_J}{kT^o}} - 1) \; e^{-\frac{x}{L_p}} \; . \tag{30}$$

D. Reverse Transient Storage Time

As mentioned in the preceding section, excess holes are injected into the n-type region during forward bias. When the reverse bias is applied, these holes remain in the vicinity of the junction and serve as current carriers in the reverse direction. For a finite period of time, the diode behaves as a short and the junction is not able to develop a reverse voltage across it.

Since the junction voltage will be positive and small in magnitude, the current will be a constant determined by the applied reverse voltage and the external circuit resistance. Hence, this period is sometimes referred to as the constant current phase of reverse recovery. Here, however, this period will be denoted as the storage time since the actual phenomena is one of removing stored charges.

At the time the concentration of excess holes at the junction becomes zero, the junction voltage becomes negative and begins to rise toward the value of the reverse applied voltage. This terminates the storage phase and the magnitude of current begins to decrease.

1. Following a steady state forward bias

Shown below is the diffusion equation with the applicable boundary conditions:

$$\frac{\partial p_{II}}{\partial T} = \frac{\partial^2 p_{II}}{\partial z^2} - p_{II} \quad , \tag{31}$$

$$p_{II}(0,\ z) = I_F e^{-z} \quad , \tag{32}$$

$$- \left. \frac{\partial p_{II}}{\partial z} \right|_{z\,=\,0} = I_R \qquad (I_R \text{ is negative}) \quad , \tag{33}$$

and

$$p_{II}\,(T,\ \infty) = 0 \ . \tag{34}$$

The first condition (Equation 32) is the carrier concentration after a steady state forward bias; the second condition (Equation 33) imposes an externally determined current I_R and the last condition (Equation 34) is the specification of long n-type material.

This equation may be solved in much the same manner as for the forward bias case. The complete solution is shown in Appendix C with the following as the result:

$$p_{II}(T,\ z) = \frac{I_R - I_F}{2} \left[e^{-z} \text{ erfc } \left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z} \text{ erfc } \left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right] + I_F\, e^{-z} \ . \tag{35}$$

This equation represents the excess hole concentration as a function of distance through the crystal (z) and time (T) of Phase II. The time T = 0 is taken as the time of application of the reverse bias.

To determine the storage time the boundary condition that $p(T_s,\ 0) = 0$ will be used. Using this condition, Equation 35 reduces to

$$0 = \frac{I_R - I_F}{2} \left[\text{erfc } (-\sqrt{T_s}) - \text{erfc } (\sqrt{T_s}) \right] + I_F \ . \tag{36}$$

In accordance with the error function identities shown earlier, the above may be written as

$$0 = (I_R - I_F) \operatorname{erf} \sqrt{T_s} + I_F \,, \tag{37}$$

which become

$$\operatorname{erf}\sqrt{T_s} = \frac{I_F}{I_F - I_R} \quad . \tag{38}$$

Using the definition of the symbols this may be written as

$$\operatorname{erf} \sqrt{\frac{t_s}{T_p}} = \frac{1}{1 + \theta} \tag{39}$$

where θ is the magnitude of I_R/I_F.

As might be expected, the length of the storage time is related to the amount of stored charge at the end of the forward bias by the I_F term, the rate of removal of this charge by (I_R) and the average lifetime of these charged carriers (T_p). This relationship is plotted in Figure 6 and tabulated in Table 3.

Obtaining the storage time is not the only consideration during this period. Since the diode cannot be considered in a steady state reverse bias until all of the excess minority carriers are removed, the remaining stored charge also must be considered.

Due to the constant current feature of this phase, the slope of the hole concentration at the junction will be a constant and determined by I_R. If the reverse current is very large, this slope will be great and the concentration will rapidly go to zero at the junction. However, in this case the carriers out in the n-type material will not have had time to diffuse

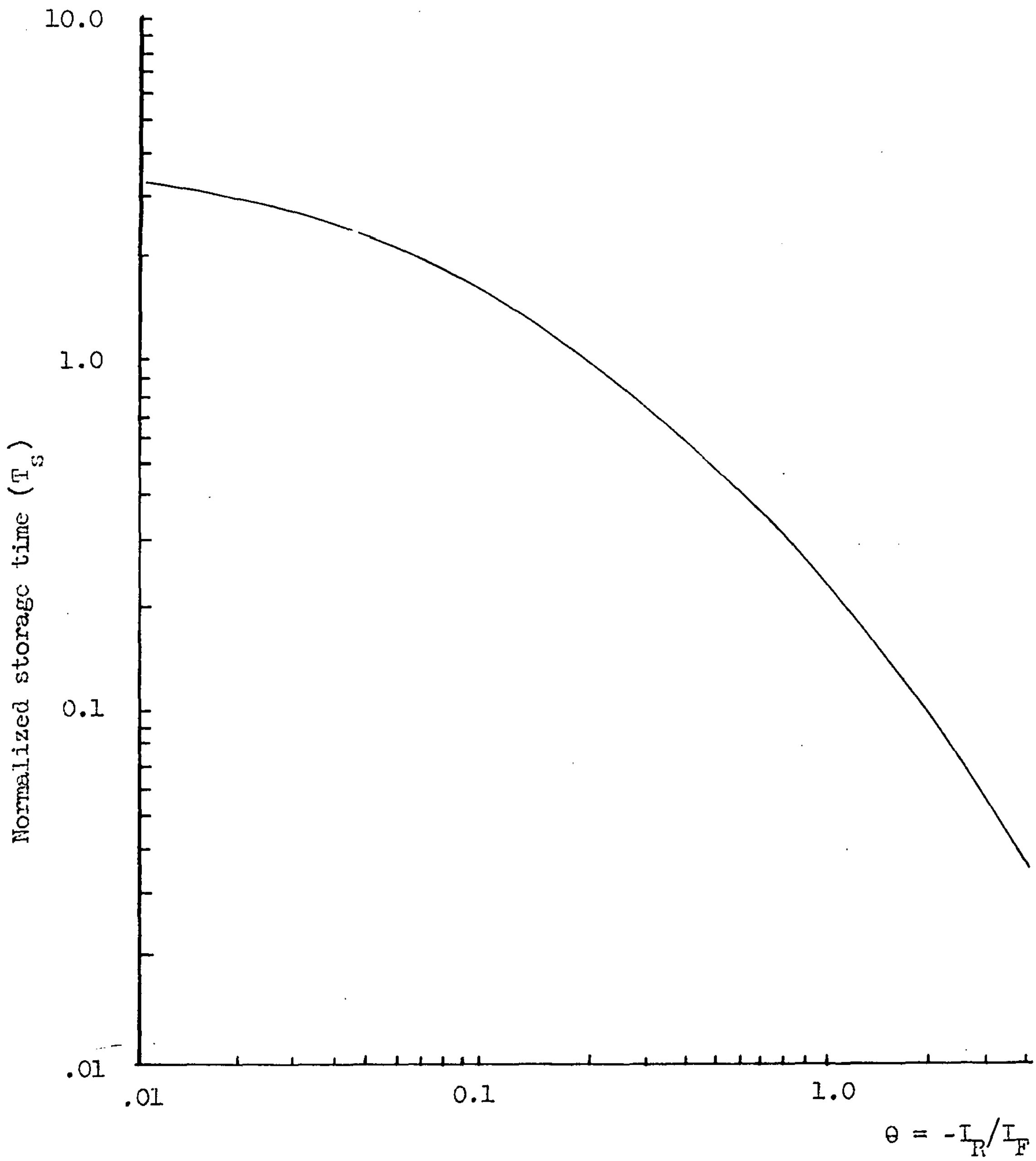


Figure 6. Storage time versus θ

Table 3. Storage time tabulation

Reference equation: $\text{erf}\ \sqrt{T_s} = \frac{1}{1 - I_R/I_F}$

$\frac{-I_F}{I_R} = \frac{1}{\theta}$	$\frac{-I_R}{I_F} = \theta$	$\frac{1}{1+\theta}$	$\sqrt{T_s}$	T_s
200.0	.005	.9950	1.998	3.95
100.0	.01	.9901	1.827	3.34
50.0	.02	.9804	1.65	2.72
40.0	.025	.9756	1.592	2.54
25.0	.04	.9615	1.463	2.14
10.0	.10	.9091	1.196	1.43
5.357	.187	.8427	1.0	1.0
5.0	.20	.8333	.978	.966
4.0	.25	.8000	.908	.824
3.0	.33	.7500	.813	.661
2.5	.40	.7143	.755	.570
2.0	.50	.6666	.684	.468
1.5	.66	.6000	.595	.354
1.0	1.0	.5000	.477	.228
.66	1.5	.4000	.3708	.1376
.5	2.0	.3333	.305	.093
.4	2.5	.2857	.259	.0672
.33	3.0	.2500	.225	.0506
.25	4.0	.2000	.1792	.03215
.20	5.0	.1667	.149	.0222
.10	10.0	.0909	.0807	.00651
.01	100.0	.0099	.009	.000081

out and a large percentage of the charge will remain at the end of the storage time.

If the reverse current is limited to a small value, the hole concentration slope at the junction will be small and consequently the storage time long. However, only a small portion of the stored charge remains at the end of the storage period.

This discussion is illustrated in Figures 7 and 8. Figure 7 shows the variation of hole concentration during the storage period. This plot is made for a storage time of $T_s = 0.2$ which corresponds to a value of $\Theta = 1.115$. The points are obtained by using the value of $-I_R/I_F = 1.115$ in Equation 35 and calculating $p(z)$ for $T = 0$, 0.05, 0.10 and 0.20.

Figure 8 indicates the amount of holes remaining in the n-type material after a storage time of 0.0, 0.05, 0.10, 0.20 and 1.0. The points were evaluated by using the corresponding values of Θ in Equation 35 and calculating $p(z)$.

2. Following a finite forward bias time

In the section on Forward Bias, an expression for the excess hole concentration was derived. If the forward bias pulse is applied for a finite time of T_F, the carrier concentration may be found by letting $T = T_F$ in Equation 27. Upon this substitution, the following is obtained:

$$p_I(T_F, z) = \frac{I_F}{2}\left[e^{-z}\,\text{erfc}\left(\frac{z}{2\sqrt{T_F}} - \sqrt{T_F}\right) - e^{z}\,\text{erfc}\left(\frac{z}{2\sqrt{T_F}} + \sqrt{T_F}\right)\right] \quad . \tag{40}$$

If the forward bias is terminated at time T_F and a reverse bias is applied, Equation 40 will become the initial concentration for the storage

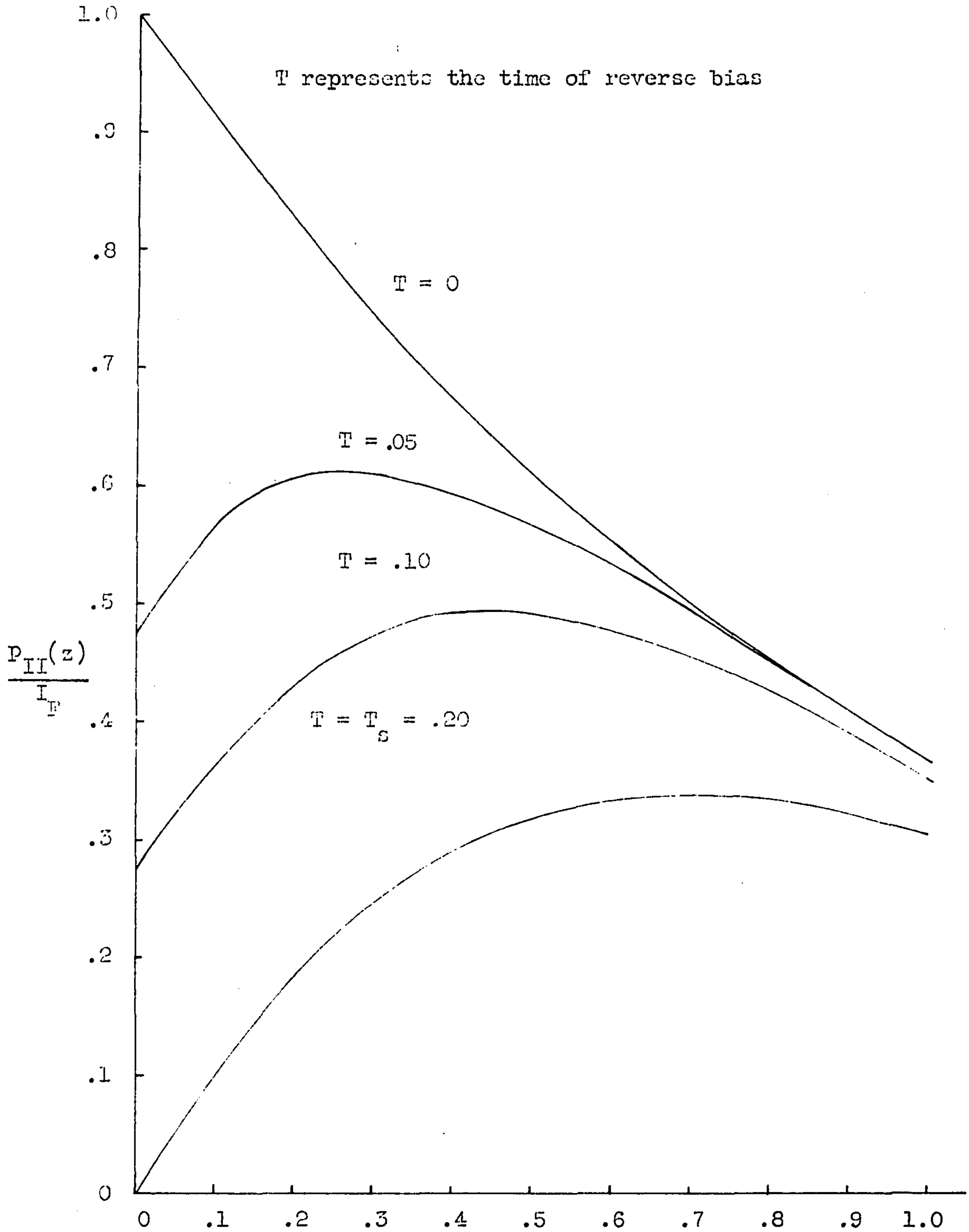


Figure 7. Minority carrier concentration during storage time

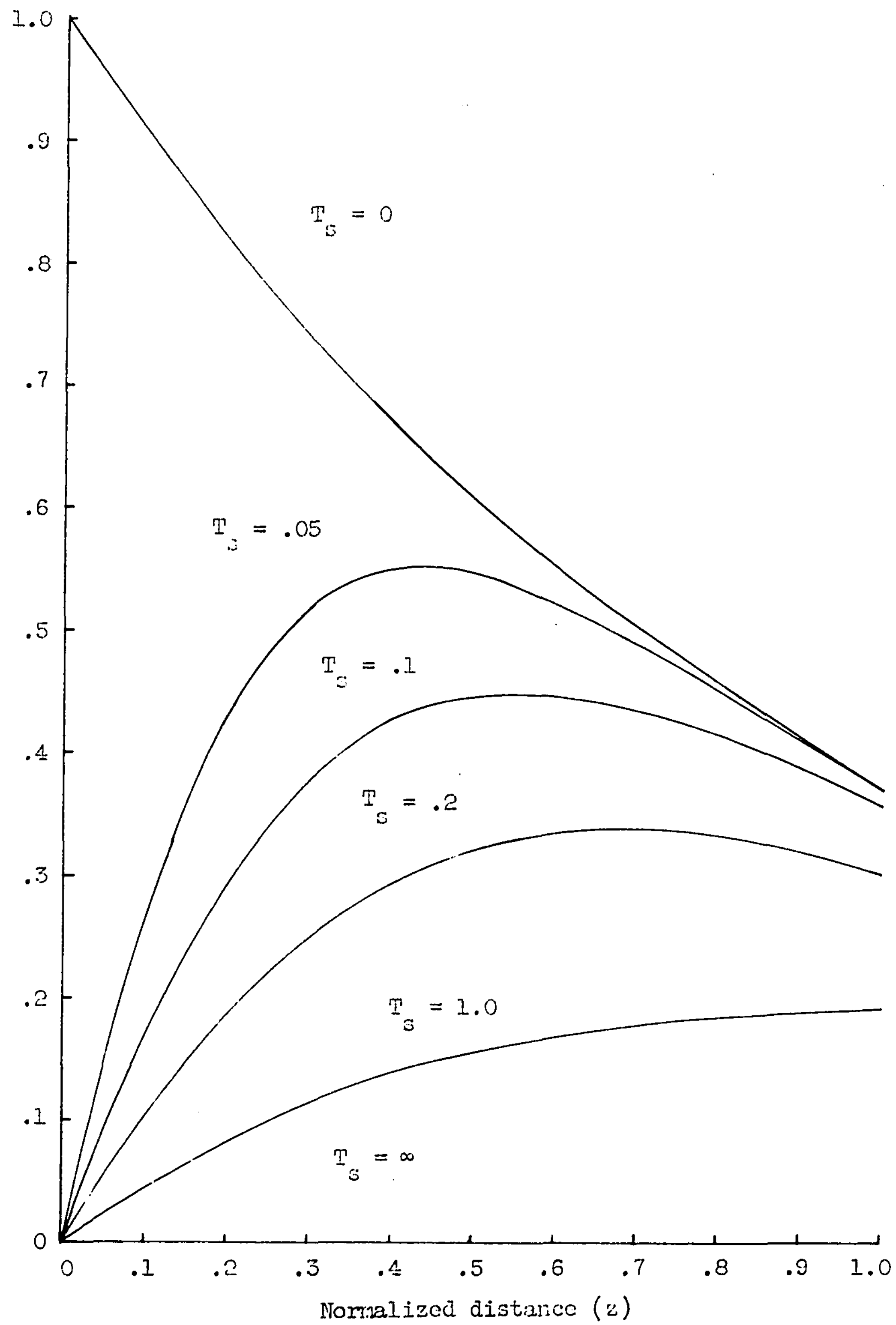


Figure 8. Remaining stored charge at the end or the storage period

phase. This may be represented as $p_I(T_F, z) = p_{II}(0, z)$.

To analyze the storage phase under this condition, Equation 31 will be solved with the boundary conditions being Equation 33, Equation 34 and the following equation:

$$p_{II}(0, z) = \frac{I_F}{2} \Big[e^{-z} \operatorname{erfc} \left(\frac{z}{2\sqrt{T_F}} - \sqrt{T_F}\right) - e^{z} \operatorname{erfc} \left(\frac{z}{2\sqrt{T_F}} + \sqrt{T_F}\right) \Big] . \tag{41}$$

These equations are the same as for the steady state forward bias case except for initial minority carrier concentration (Equation 41). This one difference, however, make the equation much more difficult to handle. Because of this, the exponential approximation will be used to replace the error functions appearing in the boundary conditions.

Whenever the approximation is employed, the general series will be used in the analysis. Once the results are obtained, the terms of the approximating series will be substituted in from Appendix A. Equation 41 then becomes

$$p_{II}(0, z) = \frac{I_F}{2} \Big[e^{-z} \Big(1 - \sum_{i=1}^{N} (-)\, C_i\, e^{(-)a_i\left(\frac{z}{2\sqrt{T_F}} - \sqrt{T_F}\right)}\Big) - e^{z} \Big(1 - \sum_{i=1}^{N} C_i\, e^{a_i\left(\frac{z}{2\sqrt{T_F}} + \sqrt{T_F}\right)}\Big) \Big] . \tag{42}$$

Where the sign in ( ) are to be used if $\sqrt{T_F} > z/2\sqrt{T_F}$. To simplify this and other expressions in this thesis, the following will be defined

$$- \sum_{i=0}^{N} C_i \, e^{a_i x} = \text{erfc } x = 1 - \text{erf } x, \text{ with } C_o = -1 \text{ and } a_o = 0 \,. \qquad (43)$$

Hence, Equation 42 may be written as

$$p_{II}(0, z) = \frac{I_F}{2} \Big[\, e^{z} \sum_{i=0}^{N} C_i \, e^{a_i \left(\frac{z}{2\sqrt{T_F}} + \sqrt{T_F}\right)}$$

$$+ \, e^{-z} \Big(1 - \sum_{i=1}^{N} (-) \, C_i \, e^{(-)a_i \left(\frac{z}{2\sqrt{T_F}} - \sqrt{T_F}\right)} \Big) \Big] \,. \qquad (44)$$

The solution to Equation 31, with boundary conditions Equations 44 and 34, may now be obtained. This complete solution is carried out in Appendix D with the following result:

$$p_{II}(T,z) = - \frac{I_F}{2} \Big\{ \sum_{i=0}^{N} (-)C_i e^{(\pm)a_i \sqrt{T_F}} \, \frac{e^{\left[\left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right)^2 - 1\right] T}}{2}$$

$$\Big[e^{-z\left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right)} \text{erfc} \left(\frac{z}{2\sqrt{T}} - \left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right) \sqrt{T}\right) - e^{z\left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right)}$$

$$\left(\text{erfc} \left[\frac{z}{2\sqrt{T}} + \left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right) \sqrt{T}\right] - 2\right) \Big] - \sum_{i=0}^{N} (-)C_i \, e^{a_i \sqrt{T_F}}$$

$$\frac{e^{\left[\left(\frac{a_i}{2\sqrt{T_F}} + 1\right)^2 - 1\right] T}}{2} \Big[e^{-z\left(\frac{a_i}{2\sqrt{T_F}} + 1\right)} \text{erfc} \left(\frac{z}{2\sqrt{T}} - \left(\frac{a_i}{2\sqrt{T_F}} + 1\right) \sqrt{T}\right)$$

$$- e^{z\left(\frac{a_i}{2\sqrt{T_F}} + 1\right)} \left(\text{erfc} \left[\frac{z}{2\sqrt{T}} + \left(\frac{a_i}{2\sqrt{T_F}} + 1\right) \sqrt{T}\right] - 2\right) \Big] \Big\} - \frac{I_R}{2} \Big[e^{-z}$$

$$\text{erfc} \left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z} \, \text{erfc} \left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right) \Big] \,. \qquad (45)$$

where the sign in ( ) is to be used for $i \geq 1$ if $\sqrt{T_F} > \frac{z}{2\sqrt{T_F}}$.

Also in Appendix D, it was shown that when the value of T_F was permitted to approach infinity, Equation 45 reduced to Equation 35 of the steady state forward bias case.

Due to the exponential terms appearing one might be concerned about the behavior of this function as $T \rightarrow \infty$. Upon consideration of this equation, it may be seen that the boundness of the function is threatened when $\sqrt{T_F} \leq \frac{z}{2\sqrt{T_F}}$. Since then

$$\left(\frac{a_i}{2\sqrt{T_F}} - 1\right)^2 - 1 > 0 \quad \text{with all a's} < 0. \tag{46}$$

We will then be concerned about terms of the following form:

$$e^{(b^2 - 1)T}\left[e^{bz} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + b\sqrt{T}\right) - e^{-bz} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - b\sqrt{T}\right)\right], \tag{47}$$

where $|b| > 1$. To analyze this expression for large values of T, the following asymoptic expressions for the complementary error function term will be used:

$$\operatorname{erfc} b\sqrt{T} \rightarrow \frac{e^{-b^2T}}{b\sqrt{\pi T}} \quad \text{for } b > 0, \tag{48}$$

and

$$\operatorname{erfc}(-b\sqrt{T}) \rightarrow 2 - \frac{e^{-b^2T}}{b\sqrt{\pi T}} \quad \text{for } b > 0. \tag{49}$$

These were derived by finding a function whose ratio to the complementary error function approached unity for large values of the argument.

It now may be seen that for a finite z and very large values of T, Equation 47 becomes

$$e^{(b^2 - 1)T} \left(\frac{e^{bz} e^{-b^2 T}}{b \sqrt{\pi T}} + \frac{e^{-bz} e^{-b^2 T}}{b \sqrt{\pi T}} \right) \quad . \tag{50}$$

Which becomes

$$\frac{e^{bz} e^{-T}}{b \sqrt{\pi T}} - \frac{e^{-bz} e^{-T}}{b \sqrt{\pi T}} \rightarrow 0 \text{ as } T \rightarrow \infty , \tag{51}$$

and hence the equation is well behaved.

The storage time may be found by letting $p_{II}(T_s, 0) = 0$. Equation 45 then reduces to

$$p(T_s, 0) = 0 = - \frac{I_F}{2} \Bigg[\sum_{i=0}^{N} C_i \, e^{(\pm)a_i \sqrt{T_F}} \, e^{\left[\left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right)^2 - 1\right] T_s}$$

$$\left(\operatorname{erf} \left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right) \sqrt{T_s} + 1\right) - \sum_{i=0}^{N} C_i e^{a_i \sqrt{T_F}} \, e^{\left[\left(\frac{a_i}{2\sqrt{T_F}} + 1\right)^2 - 1\right] T_s}$$

$$\left(\operatorname{erf} \left(\frac{a_i}{2\sqrt{T_F}} + 1\right) \sqrt{T_s} + 1\right)\Bigg] + I_R \operatorname{erf} \sqrt{T_s} \; ; \tag{52}$$

where the signs in ( ) are to be used for $i \geq 1$ if $\sqrt{T_F} > z/2 \sqrt{T_F}$.

It may be readily shown that if $T_F \rightarrow \infty$ the value of T_s is determined by the error function relationship of Equation 38, and if $T_F \rightarrow 0$ the value of T_s is also zero. These, of course, are the proper end points for T_s.

Equation 52 then may be used to calculate the storage time after a forward pulse of duration T_F. This may be done by substituting in the values of I_F, I_R and T_F. The values of the a's, C's and N are determined from Appendix A. The value of T_s may then be calculated by trial and error using computer techniques.

Although this procedure will give the proper value of T_s for a given T_F and Θ, it would be beneficial to have a more easily handled expression. For this purpose, the following equation will be used to represent Equation 40:

$$p(T_F, z) \approx m\, e^{-rz} . \tag{53}$$

To illustrate that this is a reasonable choice, Equation 40 is plotted on a natural logarithm scale in Figure 9. Also on this plot is shown the approximation.

The values of m and r for a given T_F are shown in Table 4. Although these parameters were only calculated for eight discrete values of T_F, this procedure could be used to obtain an approximation for any value of T_F.

Using this approximation, the diffusion equation may be solved with boundary conditions of Equations 53, 33, and 34. This solution is carried out in Appendix E with the result being

$$\begin{aligned} p_{II}(T, z) = \frac{I_R}{2}\left[e^{-z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right] \\ + \frac{I_F}{2} m\, e^{-(1 - r^2)T}\left[e^{rz} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + r\sqrt{T}\right)\right. \\ \left. - e^{-rz}\left(\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - r\sqrt{T}\right) - 2\right)\right] . \end{aligned} \tag{54}$$

To show that this function is well behaved for $r > 1$ and $T \to \infty$, the asymontic expressions of Equation 48 and Equation 49 were again used. When this is done the terms of concern in Equation 54 become

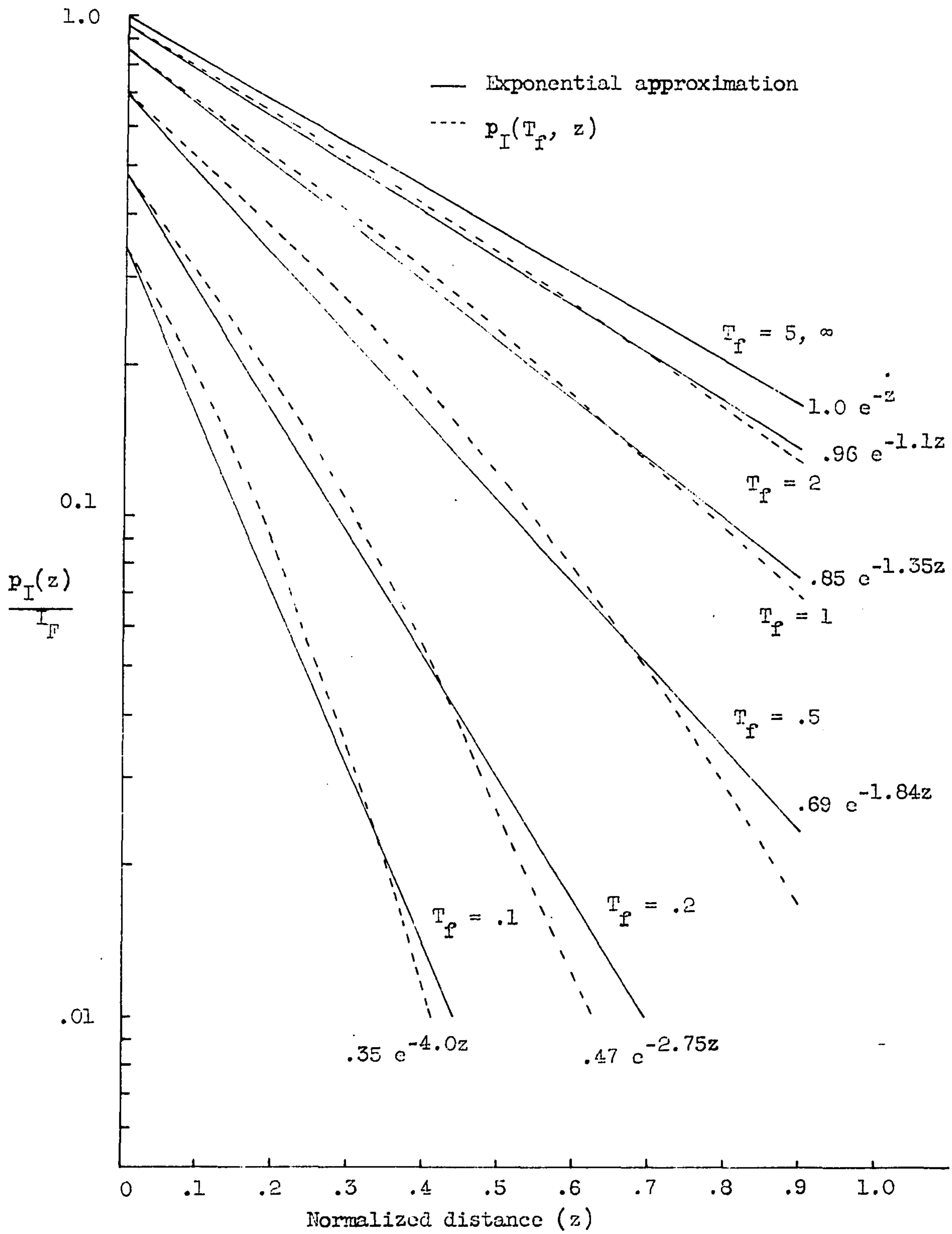


Figure 9. Exponential approximation for $p(T_f)$

Table 4. Exponential approximation for $p(T_F, z)$

Where $\frac{p}{I_F} = m\, e^{-rz}$ is an approximation for the following

$$\frac{p}{I_F} = \frac{1}{2}\left[e^{-z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right]$$

T	m	r	Approximate Maximum Error $0 \leq z \leq 1.0$
0.0	0.0	∞	00.0%
0.1	0.35	4.0	16.0%
0.2	0.47	2.75	10.0%
0.5	0.69	1.84	8.4%
1.0	0.85	1.35	4.5%
2.0	0.96	1.10	2.5%
5.0	1.0	1.0	0.5%
∞	1.0	1.0	0.0%

$$\frac{I_F}{2}\, m \left(\frac{e^{-rz}\; e^{-T}}{r\sqrt{\pi T}} + \frac{e^{rz}\; e^{-T}}{r\sqrt{\pi T}}\right) \rightarrow 0 \text{ as } T \rightarrow \infty. \qquad (55)$$

To determine the storage time, the boundary condition that $p(T_s, 0) = 0$ will be used. Imposing this condition, Equation 48 reduces to

$$I_R \operatorname{erf} \sqrt{T_s} = - m\, I_F\, e^{-(1 - r^2)T_s} (\operatorname{erfc} r\sqrt{T_s}). \qquad (56)$$

The effect of the time of forward bias upon storage time is shown in the plot of Θ versus T_s in Figure 10. The fixed parameter here being T_F. This plot was constructed by selecting a value of T_F and then finding the value of r and m from Table 4. The points then were calculated by picking a value of T_s and calculating Θ.

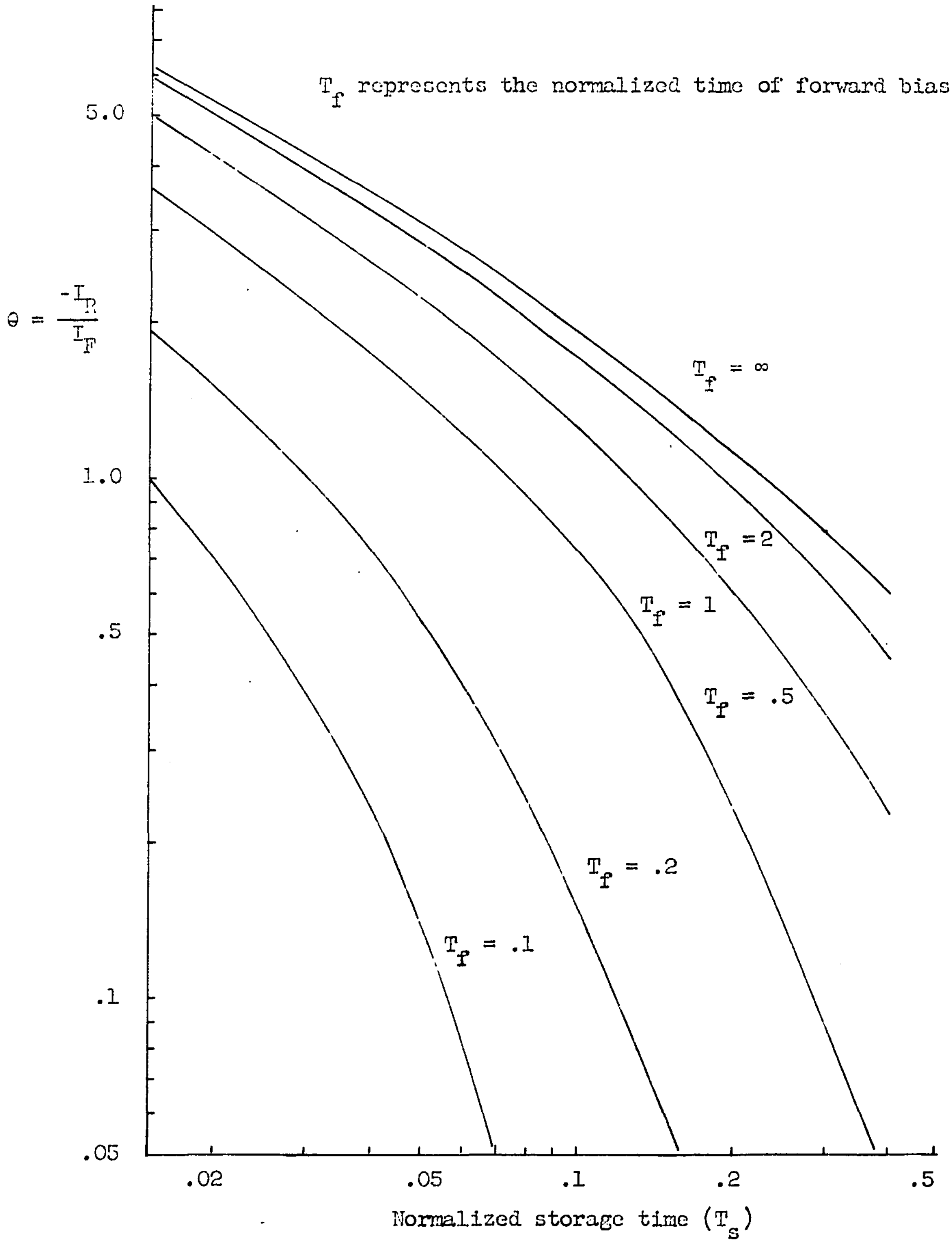


Figure 10. Storage time following a finite forward bias period

E. Reverse Recovery

At the end of the storage period, the concentration of excess minority carriers at the junction has deminished to zero. However, as shown in the previous section there remains a quantity of these carriers in the n-type region. Before the diode will reach a steady state reverse bias, this concentration must be reduced to zero throughout the entire n-type region.

The junction current does not change instantaneously from the value during the storage phase to the reverse saturation current. Instead, it decays gradually in a recovery tail. This current is made up of essentially two components. First there is the diffusion current associated with the removal of the excess hole concentration mentioned above. Second, there is a current associated with the charging of the depletion layer capacitance. The capacitance current arises from the fact that at the beginning of this phase the junction voltage is zero whereas at the end of the phase the junction voltage is the magnitude of reverse voltage in the circuit. The change in voltage across the junction capacitance, which is itself a function of the voltage, results in capacitance current.

Usually, the capacitance current is neglected and only the diffusion current considered. In this analysis, however, both of these current components will be considered on a superposition basis. In other words the diffusion component of current will first be calculated and then the corresponding magnitude of capacitance current will be investigated. If the capacitance current is comparable to the diffusion current, it may be added to obtain the total junction current.

1. Diffusion current

The diffusion current associated with the removal of the remaining stored charge will be found for the following case:

$$\frac{\partial p_{III}}{\partial T} = \frac{\partial^2 p_{III}}{\partial z^2} - p_{III} \; . \tag{57}$$

With the boundary conditions being

$$p_{III}(0, z) = \frac{I_R - I_F}{2} \left[e^{-z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T_s}} - \sqrt{T_s}\right) - e^{z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T_s}} + \sqrt{T_s}\right) \right] + I_F e^{-z}, \tag{58}$$

$$p_{III}(\infty, z) = 0 \; , \tag{59}$$

and

$$p_{III}(T, \infty) = 0. \tag{60}$$

Equation 57 is the diffusion equation for the case of no potential gradient in the n-type material.

From the first boundary condition one sees that here the analysis is being carried out for a reverse bias following a steady state forward bias. Hence this initial hole density was found by letting $T = T_s$ in Equation 35.

The second boundary condition, Equation 59, needs an explanation since it is to some extent an approximation. By definition the quantity $p(T, \bar{z})$ is the difference between the actual hole concentration and that of the intrinsic hole concentration (p_n). Since at steady state reverse bias the hole density at the junction ($z = 0$) is zero, the value of $p_{III}(\infty, 0)$ should be $-p_n$. Also the slope of p_{III} at the junction should be related to the reverse saturation current (I_s).

As shown from the boundary conditions, however, the concentration of holes has been assumed to reduce to the equilibrium value p_n and remains at this value for all z. This assumption has the effect of neglecting I_s since here the magnitude of current will reduce to zero rather than I_s. If it were desired to consider this saturation current, the magnitude of current below I_s could be set equal to I_s with very small error in the current waveshape.

For the solution to the Equations 57 through 60, the Laplace Transform and the exponential approximation to the error function was again used. This complete solution is carried out in Appendix F with the following results:

$$p_{III}(T,z) = \frac{I_R - I_F}{2} \Big[\sum_{i=0}^{N} (-)C_i e^{(\pm)a_i \sqrt{T_s}} \frac{e^{\left[\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)^2 - 1\right] T}}{2}$$

$$\Big(e^{-z\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)} \operatorname{erfc}\left[\frac{z}{2\sqrt{T}} - \left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)\sqrt{T}\right] + e^{z\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)}$$

$$\Big(\operatorname{erfc}\left[\frac{z}{2\sqrt{T}} + \left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)\sqrt{T}\right] - 2\Big)\Big) - \sum_{i=0}^{N} C_i e^{a_i \sqrt{T_s}} \frac{e^{\left[\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)^2 - 1\right] T}}{2}$$

$$\Big(e^{-z\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)} \operatorname{erfc}\left[\frac{z}{2\sqrt{T}} - \left(\frac{a_i}{2\sqrt{T_s}} + 1\right)\sqrt{T}\right] + e^{z\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)}$$

$$\Big(\operatorname{erfc}\left[\frac{z}{2\sqrt{T}} + \left(\frac{a_i}{2\sqrt{T_s}} + 1\right)\sqrt{T}\right] - 2\Big)\Big)\Big] - \frac{I_F}{2}\left[e^{-z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right)\right.$$

$$\left. + e^{z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right] + I_F e^{-z} \qquad (61)$$

where the signs in () are to be used when $\sqrt{T_s} > \frac{z}{2\sqrt{T_s}}$ and $i \geq 1$.

Upon examination of this equation one sees that there exist exponential terms which at times have positive exponent containing the variable T or z. To show that the above equation is well behaved for all values of T and z each set of terms were checked for boundness in Appendix F.

Although the holes density is of interest, the magnitude of reverse current flowing during this phase is of greater interest. This may be obtained by performing the following:

$$I(T) = - \left. \frac{\partial p_{III}}{\partial z} \right|_{z=0} . \tag{62}$$

This calculation is carried out in Appendix F. It is of interest to compare the case of $T_s = 0$ with that derived by others. For this condition, $\sqrt{T_s} < z/2\sqrt{T_s}$ and Equation F-15 reduces to

$$I_{III}(T) = - I_F \left[\operatorname{erf} \sqrt{T} + \frac{e^{-T}}{\sqrt{\pi T}} - 1\right] , \tag{63}$$

which agrees with that derived in two cited papers (6, 13) and with a similar expression, which includes I_s, derived by B. Lax and S. F. Neustadter (8).

For the general case of T_s greater than zero, the current during Phase III is given by

$$I_{III}(T) = (I_R - I_F) \left\{ \sum_{i=1}^{N} -C_i e^{a_i \sqrt{T_s}} e^{\left[\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)^2 - 1\right]T} \left[\frac{e^{-\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)^2 T}}{\sqrt{\pi T}} + \left(\frac{a_i}{2\sqrt{T_s}} + 1\right)\left(\operatorname{erf} \left(\frac{a_i}{2\sqrt{T_s}} + 1\right)\sqrt{T}\right)\right] + I_R - I_F\left(\frac{e^{-T}}{\sqrt{\pi T}} + \operatorname{erf}\sqrt{T}\right). \tag{64}$$

For assurance of the validity of this equation, the end points may be checked. For $T = 0$, it may be shown that $I_{III}(0) = I_R$ and for $T = \infty$, $I_{III} = 0$. These are the proper boundary conditions.

For a steady state forward bias, the entire current waveshape has now been determined. Hence, for a given value of I_F and I_R, the storage time may be determined by Equation 38 and the current waveshape during the recovery phase by Equation 64. These equations are illustrated in Figure 11.

2. Junction capacitance considerations

Since the current considered here is in the same direction as the diffusion current, one can, in effect consider a capacitance placed in parallel with the diode. This consideration is illustrated in Figure 12a. When such a model is used, the following equations may be written for the circuit:

$$V = (I_C + I_J)\,R + v_J\ , \tag{65}$$

and

$$I_C = C\,\frac{dv_J}{dt} + v_J\,\frac{dC}{dt}\ . \tag{66}$$

Where V is the externally applied reverse voltage present in the circuit, I_J is the junction current due to diffusion and I_C is the current due to the creation of the space charge layer. To investigate the validity of neglecting I_C, its maximum value will be found and compared to the total diode current. The maximum capacitance current will occur when C and dv_J/dt are a maximum. These in turn will occur when $v_J = 0$.

For these calculations the equation for v_J, derived by several authors

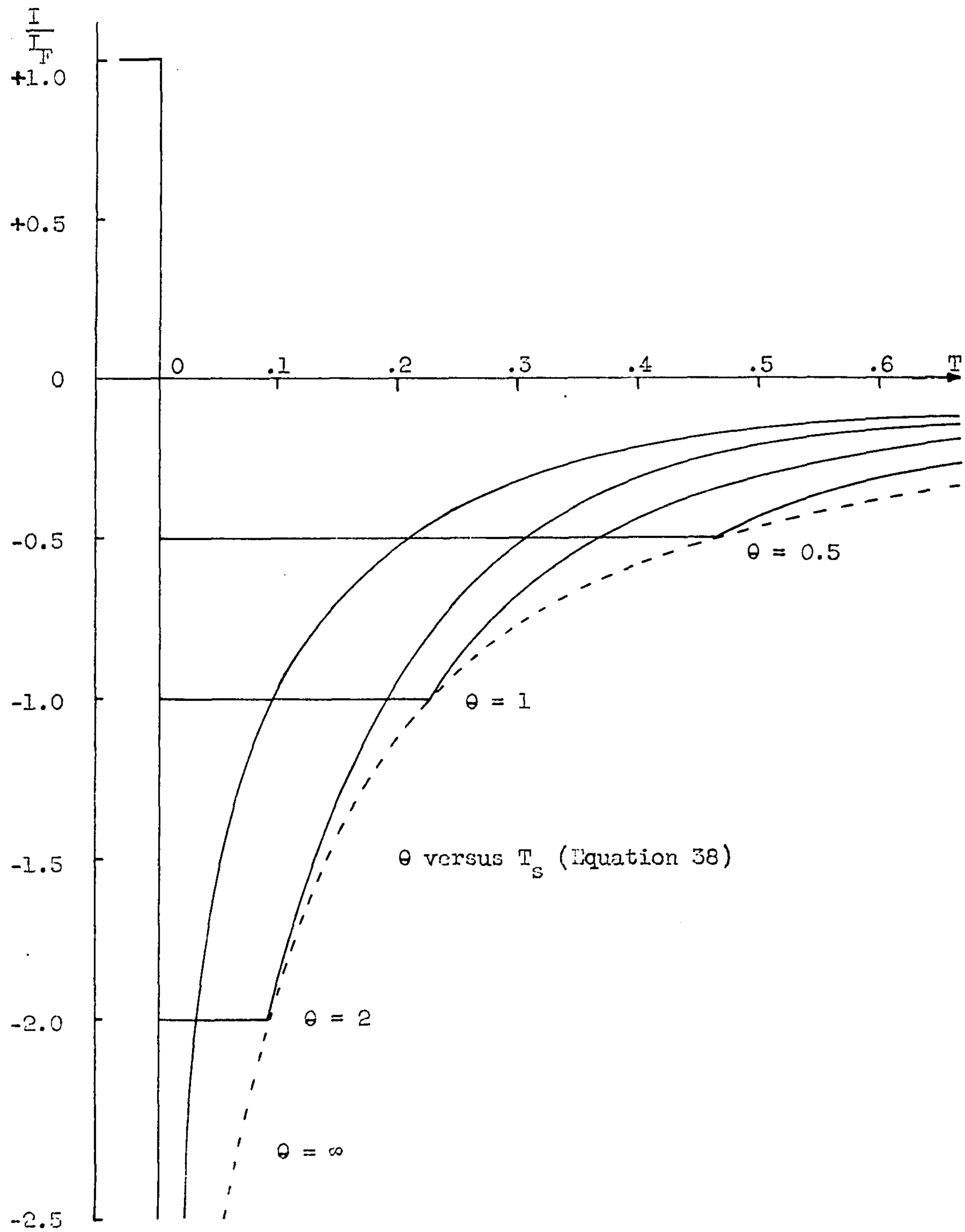
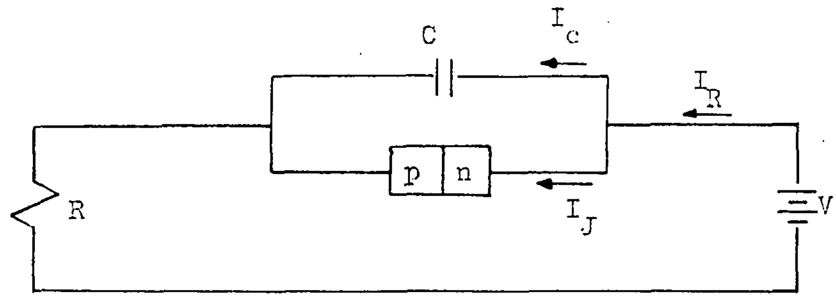
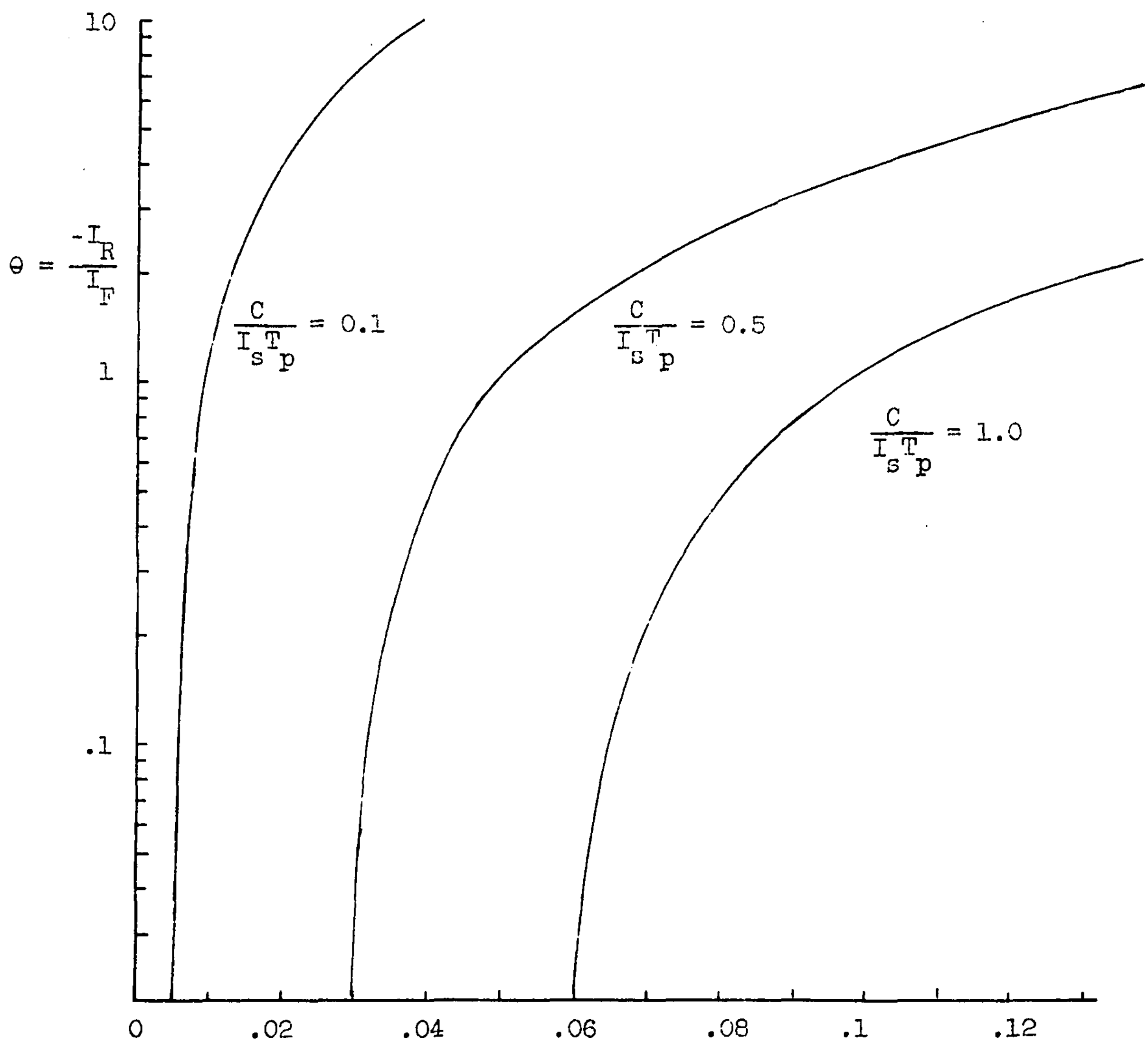


Figure 11. Reverse bias current following a steady state forward bias



(a) Equivalent circuit



(b) θ versus I_c/I_R

Figure 12. Capacitance current consideration

will be used (4, 6). This equation is

$$v_J = \frac{kT^o}{q} \; \ell n \left(1 + \frac{I_F}{I_s} - \frac{I_F - I_R}{I_s} \operatorname{erf} \sqrt{T}\right). \tag{67}$$

Which gives

$$\frac{dv_J}{dt} = \frac{kT^o}{q} \; \frac{(I_R - I_F) \;\; 2\, e^{-T}}{T_p \sqrt{\pi T} \left(I_s + I_F + (I_R - I_F) \operatorname{erf} \sqrt{T}\right)} \; . \tag{68}$$

But $v_J = 0$ when $T = T_s$; and with the aid of Equation 38 the ratio of maximum capacitance current to total current is given by

$$\frac{I_C}{I_R} = \frac{C}{T_p I_s} \; \frac{kT^o}{q} \; \frac{(I_R - I_F)\, 2\, e^{-T_s}}{I_R \sqrt{\pi \, T_s}} \; . \tag{69}$$

To illustrate the validity of neglecting I_C, Equation 69 has been plotted in Figure 12b. For this plot the three parameters of the diode (i.e., C, I_s and T_p) were lumped together as the fixed parameter of the plot.

From this figure one can see that if $C \geq I_s T_p$; the capacitance current may be neglected with little error. This specification is true for most computer and switching diodes on the market today. However, due to recent methods of decreasing the minority carrier lifetime (T_p) some of the faster diodes have a capacitance current which is a greater portion of the reverse current. To cite an example; the HD 2967, which is advertised by Hughes Semiconductor as an ultra fast switching diode, has capacitance of 4 μμf, I_s of 40×10^{-6} amps and a T_p of 26.4×10^{-9} sec (5). With these values C is approximately $4 I_s T_p$. Hence for a θ of 1 the ratio of I_C/I_R would be approximately 0.4.

F. Potential Gradient in a Finite Length N-Type Region

As derived previously in this thesis, the following equation governs the distribution of holes in the n-type semiconductor material:

$$\frac{\partial p}{\partial T} = \frac{\partial^2 p}{\partial z^2} - f\, L_p \frac{\partial p}{\partial z} - p, \text{ with } f = \frac{qE_n}{kT^o} . \tag{70}$$

In this equation E_n represents the potential gradient to which the holes are subjected.

In the previous analysis f was taken to be zero since no field was considered away from the junction. Furthermore, the n-type region will be considered to have an ohmic contact with an arbitrary surface recombination velocity (S) at a distance W from the junction. The mathematical model under consideration in this section is shown in Figure 13a.

For the steady state forward bias condition, $\partial p/\partial T = 0$ in Equation 70 and the boundary conditions become:

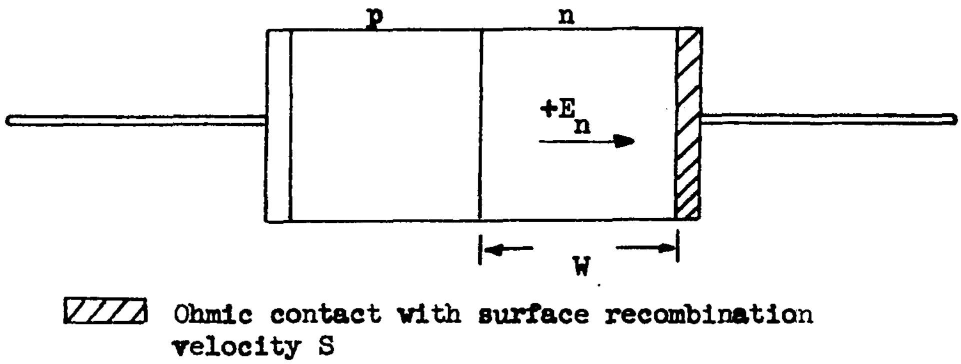
$$\frac{dp}{dz} - f\, L_p\, p = - I_F \text{ at } z = 0 \tag{71}$$

and

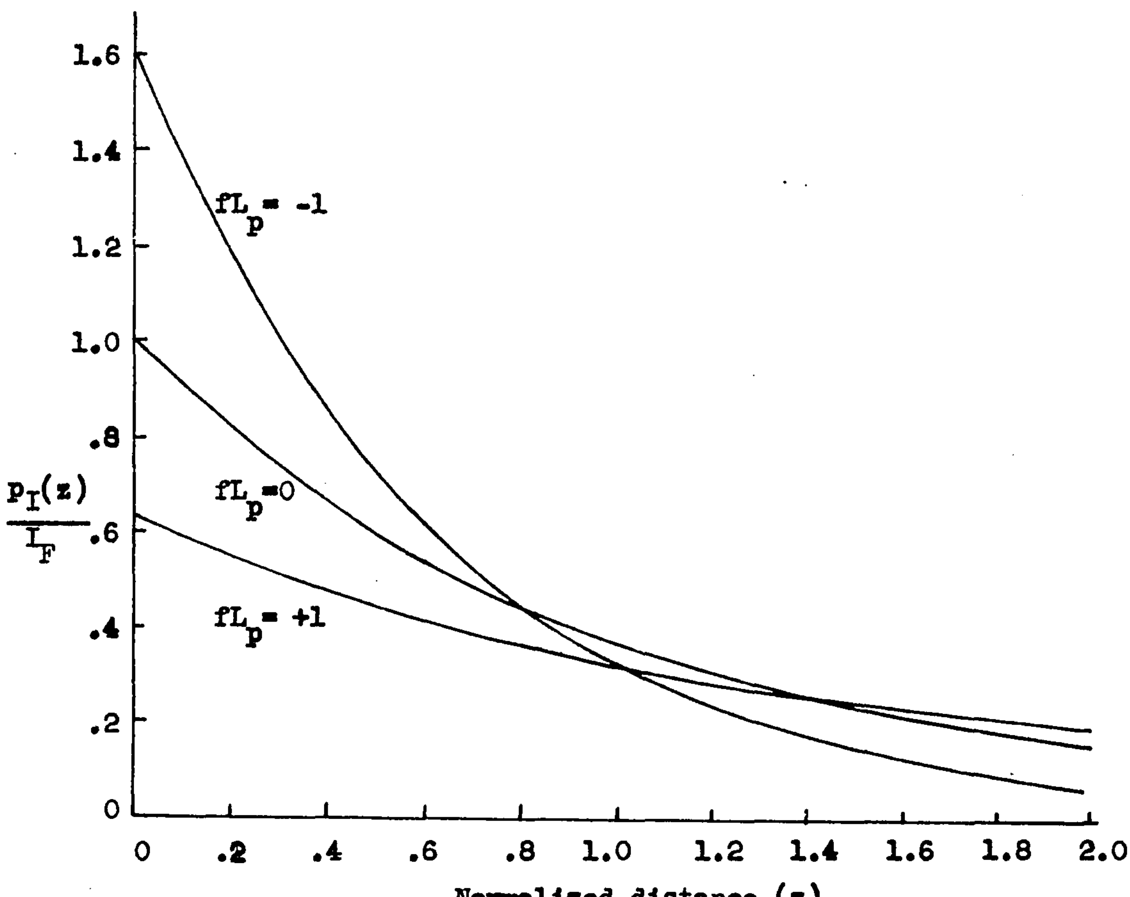
$$\frac{dp}{dz} + \left(\frac{SL_p}{D_p} - fL_p\right) p = 0 \text{ at } z = W . \tag{72}$$

Equation 71 specifies the forward current (I_F) and Equation 72 states that the diffusion current plus the drift current must equal the recombination current at the ohmic contact.

The complete solution to this set of equation is shown in Appendix G with the result being:



(a) Mathematical model



(b) Carrier density in n-type region

Figure 13. Diode with a drift field

$$p(z) = - I_F \frac{\left(\frac{fL_p}{2} - \sqrt{1 + \frac{f^2L_p^2}{4}} - \frac{SL_p}{D_p}\right) e^{\left(\frac{fL_p}{2} - \sqrt{1 + \frac{f^2L_p^2}{4}}\right) z}}{\left(\frac{fL_p}{2} + \sqrt{1 + \frac{f^2L_p^2}{4}}\right)\left(\sqrt{1 + \frac{f^2L_p^2}{4}} - \frac{fL_p}{2} + \frac{SL_p}{D_p}\right)}$$

$$\frac{+ \left(\frac{SL_p}{D_p} - \frac{fL_p}{2} - \sqrt{1 + \frac{f^2L_p^2}{4}}\right) e^{\left(\frac{fL_p}{2}\right)z + \sqrt{1 + \frac{f^2L_p^2}{4}}(z - 2W)}}{+ \left(\frac{fL_p}{2} + \sqrt{1 + \frac{f^2L_p^2}{4}}\right)\left(\frac{fL_p}{2} + \sqrt{1 + \frac{f^2L_p^2}{4}} - \frac{SL_p}{D_p}\right) e^{-2W\sqrt{1 + \frac{f^2L_p^2}{4}}}} . \qquad (73)$$

For purposes of comparison to that derived previously, the long n-type region will be considered. When W is allowed to become very large, Equation 73 reduces to

$$p(z) = I_F \frac{e^{\left(\frac{fL_p}{2} - \sqrt{1 + \frac{f^2L_p^2}{4}}\right) z}}{\sqrt{1 + \frac{f^2L_p^2}{4}} + \frac{fL_p}{2}} . \qquad (74)$$

The dimensionless quantity $p(z)/I_F$ is plotted in Figure 13b for $fL_p = +1.0$, 0.0, and -1.0. From this figure, it may be seen that for the negative drift field has a larger hole concentration at the junction but a faster decay rate. This is because the negative field tends to force the holes back toward the junction, whereas, the positive field aids the hole flow for forward bias.

For the finite length n-type region, all factors in Equation 73 must be considered. This case is plotted in Figure 14 for $SL_p/D_p = 1$, 10, and 100 with $fL_p = +1.0$, 0.0, and -1.0. For this plot the value of W was taken as 1.0. Hence, the actual length of the n-type region is equal to the diffusion length of holes. From this plot one sees that the ohmic surface recombination velocity (S) has its greatest influence when the drift field is negative. This is because there will be a larger concentration near the ohmic contact to be influenced by this parameter.

For the reverse bias phase the storage time will be calculated for two cases. First, the storage time will be found for no drift field in a finite length n-type region. Second, a long n-type region with a specified drift field will be considered.

1. Storage time considering no drift field

To obtain an usable solution for T_s, the value of W will be specified as greater than or equal to 0.5. The storage time calculations are carried out in Appendix G with the following result:

$$\operatorname{erf} T_s = \frac{\dfrac{1 + SL_p/D_p + (1 - SL_p/D_p)\, e^{-2W}}{1 + SL_p/D_p - (1 - SL_p/D_p)\, e^{-2W}}}{1 + \theta} \tag{75}$$

Specifying $SL_p/D_p = 10$, this equation is plotted in Figure 15 for W = 0.5, 1.0, 2.0 and infinity. One interesting thing seen here is that for a specified value of W there is a limiting value of T_s. This results from the fact that the recombination at the contact would reduce the hole concentration at the junction; even if the diode were open circuited.

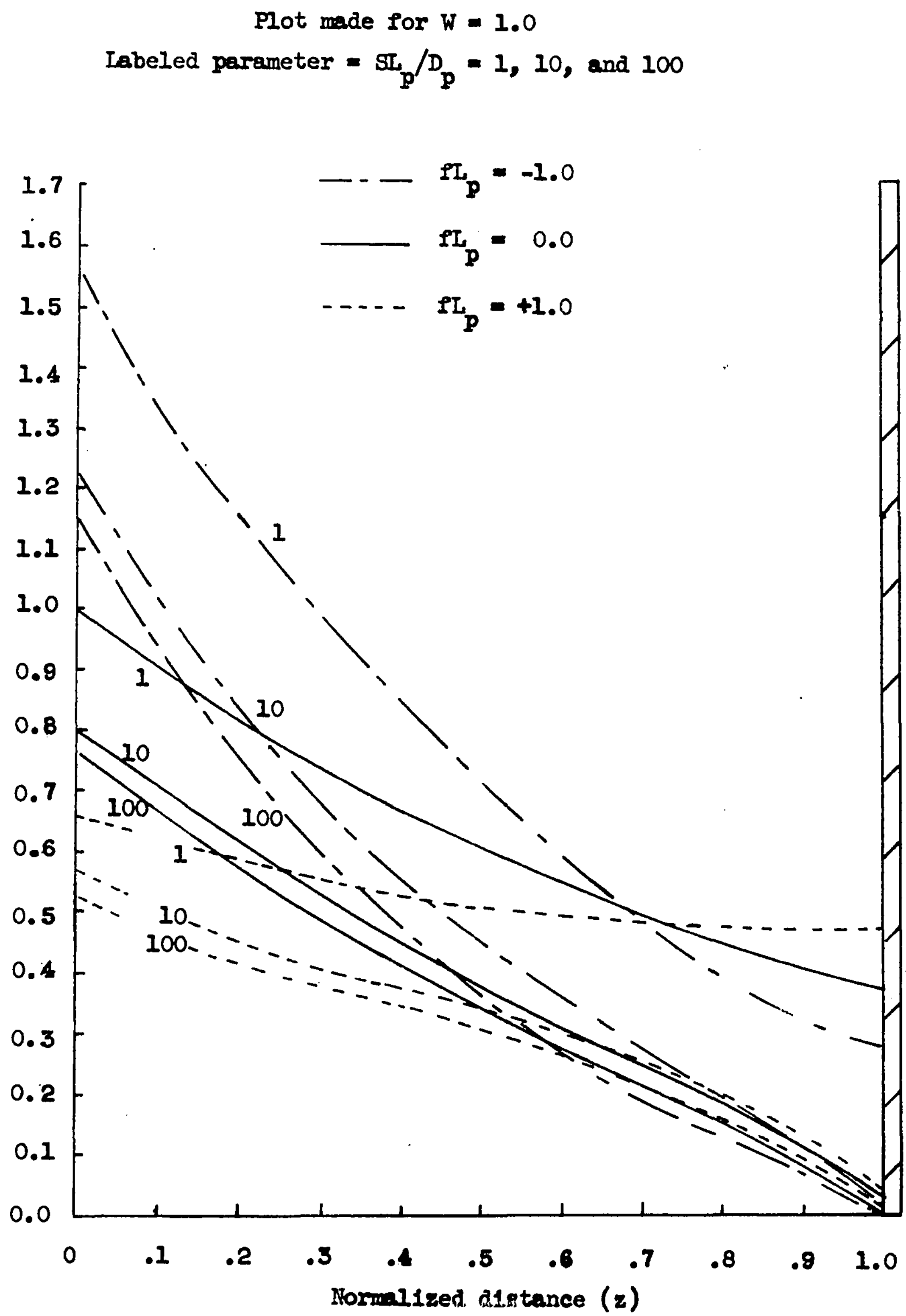


Figure 14. Finite length n-type region with a drift field

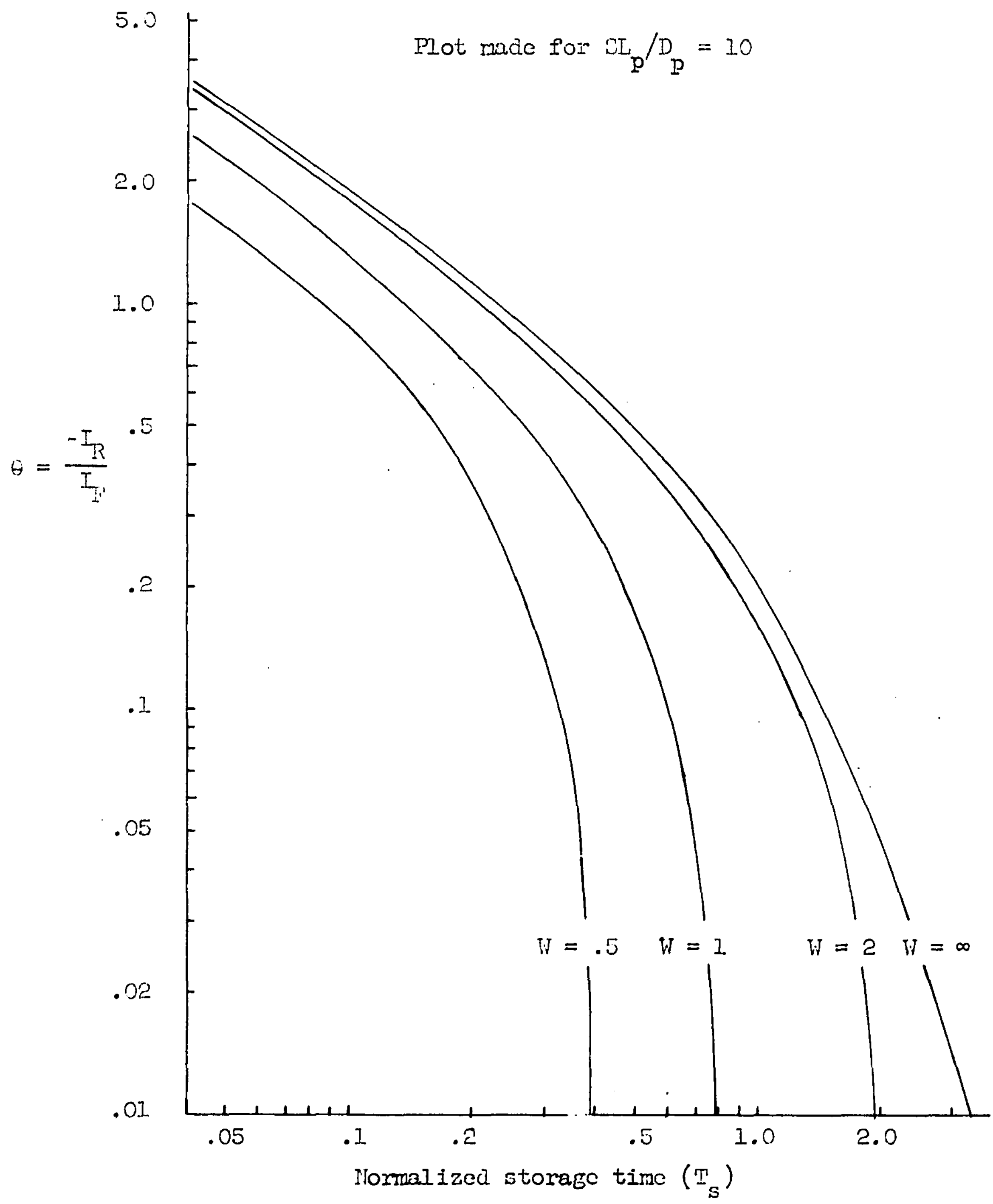


Figure 15. Storage time versus θ for a finite length n-type region

2. Storage time for a long n-type region

For a long n-type region with a drift field, the storage time is calculated in Appendix G and given by:

$$\frac{1}{1+\theta} = \left(\frac{fL_p}{2} + \sqrt{1 + \frac{f^2L_p^2}{4}}\right) \left[\frac{fL_p}{2} \left(e^{-T_s} - 1\right) + \sqrt{1 + \frac{f^2L_p^2}{4}}\right.$$

$$\left. \operatorname{erf} \sqrt{\left(1 + \frac{f^2L_p^2}{4}\right) T_s} - \frac{fL_p}{2}\, e^{-T_s} \operatorname{erf} \frac{fL_p}{2} \sqrt{T_s}\right]. \qquad (76)$$

From this equation a plot of θ versus T_s is shown in Figure 16. This figure shows that the storage time for the negative field is greater than for the no-field or negative field condition. This difference, however, is not as great as one might expect. This is due to the fact that even though there is a larger concentration of holes at the junction for the negative field case, this field aids in the removal of these holes. The opposite is true for the positive field case.

Although the storage time is greater for the negative field, only a small hole concentration will remain in the n-type region at the end of the storage period. For the positive field, T_s is less but a large portion of the hole concentration remains. Hence, the magnitude of current at the end of the storage phase will rapidly decrease for the negative field but slowly reduce for the positive field case.

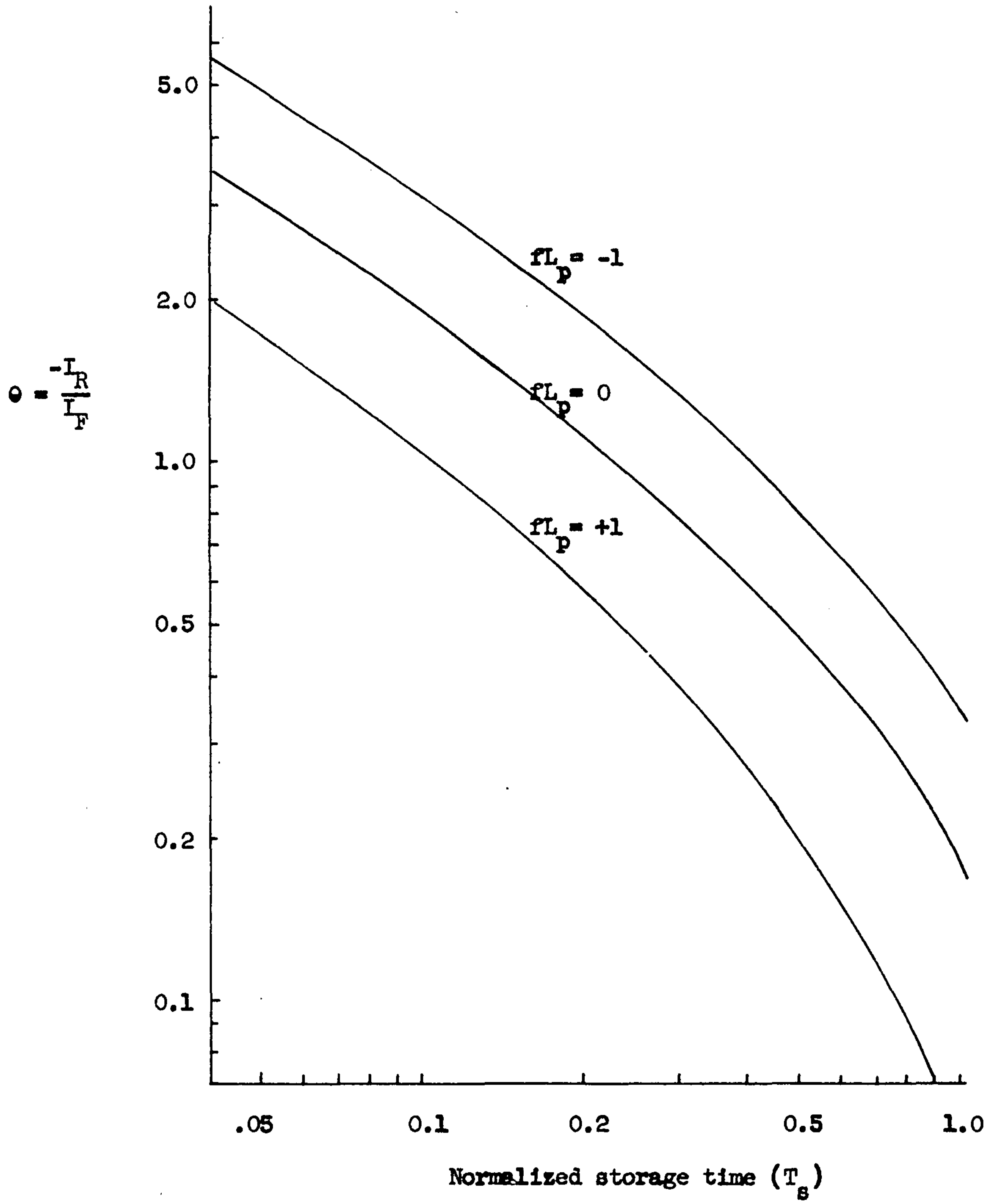


Figure 16. Storage time versus Θ for a drift field in the n-type region

V. DISCUSSION

The first mathematical model considered in this thesis was a diode with a long n-type region and no drift field. The minority carrier density and diffusion current were found for the entire switching transient. During forward bias the current is equal to I_F and the hole density is given by Equation 27. For the storage time the current is indicated by I_R and the carrier density by Equation 35. And finally for the recovery phase, the current is given by Equation 63 and the hole density by Equation 61. With this material available, the circuit designer can predict the time (after reverse bias) for a diode to reduce the magnitude of reverse current to a specified amount.

The storage time was also determined for both the steady state and finite forward bias time. For the finite forward bias time an exact solution and a more easily handled approximation were obtained.

The diode with a drift field in a finite length n-type region was then considered. Both the storage time and hole density were calculated for this model. The relation of storage time to the length of the n-type region is given by Equation 75 and to the drift field is given by Equation 76.

This author feels that the most significant contributions of this thesis are: 1. The acquisition of an equation for the current following the storage phase, 2. The consideration of the reverse bias following a finite forward bias pulse, and 3. The investigation of a finite length n-type region with a drift field.

There are primarily three limitations to the application of this material. First, an exponential approximation was used for the error function in the derivation of this material. Second, a one dimensional flow across the junction was assumed. Third, a constant lifetime of minority carriers was assumed.

The error incurred in the exponential approximation can be made very small by increasing the number of terms in the series. For the four term series used for illustration in this thesis, this error was less than five percent for most values of the argument.

The assumption of one dimensional flow will only be in jeopardy when the diode being considered has a small cross-sectional area at the junction. There will then be a component of recombination current directed toward the surface. However, it would be difficult to make a general analysis considering this surface recombination. This is because a separate analysis would have to be conducted for each individual geometry considered.

The variation of minority carrier lifetime was discussed earlier in this thesis and in general can be considered a second order effect.

With these reservations, the material of this thesis can be used to predict the reverse response of diodes for the various cases considered herein.

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VII. ACKNOWLEDGEMENTS

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VIII. APPENDIX

A. Error Function Approximation

The error function is to be approximated by the following closed series

$$\text{erf } x \approx C_1 e^{a_1 x} + C_2 e^{a_2 x} + \cdots\cdots C_n e^{a_n x} . \quad \text{(A-1)}$$

This is equivalent to

$$\text{erf } x \approx C_1 u_1^{\,x} + C_2 u_2^{\,x} + C_3 u_3^{\,x} \cdots\cdots C_n u_n^{\,x} ; \quad \text{(A-2)}$$

where $u_k = e^{a_k}$.

The error function will be equated to the approximation at N equally spaced points i.e., $x = x_0, x_1, 2x_1, 3x_1 \ldots (N - 1)x_1$. Since Equation A-2 is to be satisfied at these values of the argument, the following equations will necessarily be true:

$$
\begin{aligned}
&C_1 + C_2 + C_3 \cdots\cdots\cdots C_n = \text{erf } x_0 , \\
&C_1 u_1 + C_2 u_2 + C_3 u_3 \cdots\cdots C_n u_n = \text{erf } x_1 , \\
&C_1 u_1^{\,2} + C_2 u_2^{\,2} + C_3 u_3^{\,2} \cdots\cdots C_n u_n^{\,2} = \text{erf } 2x_1 , \quad \text{(A-3)} \\
&\quad \vdots \\
\text{and} \quad &C_1 u_1^{\,N-1} + C_2 u_2^{\,N-1} + C_3 u_3^{\,N-1} \cdots C_n u_n^{\,N-1} = \text{erf } (N - 1)\, x_1 .
\end{aligned}
$$

If the values of $u_1 \ldots u_n$ were assumed, or preassigned, this set would comprise N linear equations of n unknowns $C_1 \ldots C_n$ and could be solved exactly for N = n or approximately by the least squares method for N > n.

However, in general the u's are to be determined and at least 2n equations are needed. A further difficulty is that the equations are nonlinear in the u's. To overcome this difficulty, a substitution will be made which will result in two sets of n linear equations. (4)

In this procedure u_1, u_2, . . . u_n will be the roots of the algebraic equation

$$u^n - \alpha_1 u^{n-1} - \alpha_2 u^{n-2} \, . \, . \, . \, . \, . \, \alpha_{n-1} u - \alpha_n = 0 \, . \tag{A-4}$$

The left hand side of A-4 being identified with $(u - u_1)(u - u_2)(u - u_3)$ .....$(u - u_n)$. It remains now to obtain the α coefficients. To do this the first equation of A-3 is multiplied by α_n, the second equation by α_{n-1}, ... the nth equation by α and the (n + 1)th by -1 and the results added. In view of the fact that all u's satisfy A-4, the result will be as follows:

$$\text{erf } nx_1 - \alpha_1 \text{ erf } (n - 1)x_1 \, . \, . \, . \, \alpha_{n-1} \text{ erf } x_1 - \alpha_n \text{ erf } x_0 = 0. \tag{A-5}$$

A set of N-n-1 additional equations of similar type may be obtained in the same way by starting successively with the second, third, . . . (N-n)th equations. Following this procedure, we see that Equations A-3 and A-4 result in the following set of equations:

$$\begin{aligned} &\alpha_1 \text{ erf } (n-1)x_1 + \alpha_2 \text{ erf } (n-2)x_1 + .. \, \alpha_n \text{ erf } (x_0) = \text{erf } nx_1 \, , \\ &\alpha_1 \text{ erf } nx_1 + \alpha_2 \text{ erf } (n-1)x_1 + . \, . \, \alpha_n \text{ erf } (x_1) = \text{erf } (n + 1)x_1, \\ &\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ \text{and } &\alpha_1 \text{ erf } (N-2)x_1 + \alpha_2 \text{ erf } (N -3)x_1 \, . \, . \, \alpha_n \text{erf } (N-n-1)x_1 = \text{erf}(N-1)x_1. \end{aligned} \tag{A-6}$$

The values of the error functions may be found from tables and the above equations solved exactly for $N = 2n$ or solved approximately for $N > 2n$.

The coefficients of the factorable Equation A-4 have now been found. Upon factoring this equation, the values of $u_1, u_2, \ldots u_n$ will be known. These values then may be substituted back into Equation A-3 and the n equations solved for the C's. Since the a's may be found from the u's, Equation A-1 is now completely determined.

B. Forward Bias

The solution to the following differential equation is to be obtained

$$\frac{\partial p_I}{\partial T} = \frac{\partial^2 p_I}{\partial z^2} - p_I \quad . \tag{B-1}$$

The boundary conditions being

$$p(0, z) = 0 \ , \tag{B-2}$$

$$- \left. \frac{\partial p_I}{\partial z} \right|_{z = 0} = I_F \ , \tag{B-3}$$

and

$$p\,(T, \infty) = 0 \ . \tag{B-4}$$

To solve Equation B-1 subject to the above boundary conditions, the Laplace Transform will be used. Designating the transformed dependent variable by a capital and letting s be the independent variable of the transform the following is obtained:

$$sP = \frac{d^2 P}{dz^2} - P \ , \tag{B-5}$$

$$- \left. \frac{dP}{dz} \right|_{z = 0} = \frac{I_F}{s} , \qquad \text{(B-6)}$$

and

$$P(s, \infty) = 0 . \qquad \text{(B-7)}$$

The solution of this equation has the following form:

$$P(s, z) = C_1 e^{z\sqrt{1 + s}} + C_2 \, e^{-z\sqrt{1 + s}} . \qquad \text{(B-8)}$$

From boundary condition B-7, C_1 is found to equal zero and from the condition B-6, C_2 is given by

$$C_2 = \frac{I_F}{s\sqrt{1 + s}} . \qquad \text{(B-9)}$$

Substituting these back into Equation B-8 one obtains

$$P(s, z) = \frac{I_F}{s\sqrt{1 + s}} \; e^{-z\sqrt{1 + s}} . \qquad \text{(B-10)}$$

The inverse transformation of this equation may be found with the aid of pair #825 given by Campbell and Foster (3). Using this transformation, p(T, z) becomes

$$p(T, z) = \frac{I_F}{2} \left[\; e^{-z} \operatorname{erfc} \left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z} \operatorname{erfc} \left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right] . \qquad \text{(B-11)}$$

C. Storage Period Following Steady State Forward Bias

The solution to the following differential equation is to be obtained

$$\frac{\partial p_{II}}{\partial T} = \frac{\partial^2 p_{II}}{\partial z^2} - p_{II} . \qquad \text{(C-1)}$$

The boundary conditions during this period being

$$p(0, z) = I_F e^{-z} , \tag{C-2}$$

$$- \left. \frac{dp}{dz} \right|_{z = 0} = I_R , \tag{C-3}$$

and

$$p(T, \infty) = 0. \tag{C-4}$$

Taking the Laplace Transformation and denoting the dependent variable with capitals, one obtains

$$\frac{d^2P}{dz^2} - P(1 + s) = - I_F e^{-z} , \tag{C-5}$$

$$- \left. \frac{dP}{dz} \right|_{z = 0} = \frac{I_R}{s} , \tag{C-6}$$

and

$$P(s, \infty) = 0 . \tag{C-7}$$

The solution to this equation is of the following form

$$P(s, z) = C_1' e^{z \sqrt{1 + s}} + C_2' e^{-z \sqrt{1 + s}} - \frac{I_F}{s} e^{-z}. \tag{C-8}$$

From the boundary conditions C_6 and C_7, C_1' is found to be equal to zero and C_2' is given below:

$$C_2' = (\frac{I_R}{s} - \frac{I_F}{s}) \frac{1}{\sqrt{1 + s}} . \tag{C-9}$$

Hence,

$$P(s, z) = \frac{I_R - I_F}{s \sqrt{1 + s}} e^{-z \sqrt{1 + s}} - \frac{I_F}{s} e^{-z} . \tag{C-10}$$

It may be recognized that the first term on the right is of the same form as that solved in Appendix B and the second has a well known inverse transform. Hence, in the time domain this equation becomes

$$p(T, z) = \frac{I_R - I_F}{2}\left[e^{-z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right] + I_F e^{-z}. \quad \text{(C-11)}$$

D. Storage Period Following Finite Forward Bias Time

The following equation is to be solved

$$\frac{\partial p_{II}}{\partial T} = \frac{\partial^2 p_{II}}{\partial z^2} - p_{II} \quad . \quad \text{(D-1)}$$

The boundary conditions being

$$p_{II}(0, z) = \frac{I_F}{2}\left(e^{z} \sum_{i=0}^{N} C_i e^{a_i\left(\frac{z}{2\sqrt{T_F}} + \sqrt{T_F}\right)} - e^{-z} \sum_{i=0}^{N} (-)C_i e^{(-)a_i\left(\frac{z}{2\sqrt{T_F}} - \sqrt{T_F}\right)}\right) \quad \text{(D-2)}$$

$$-\left.\frac{\partial p_{II}}{\partial z}\right|_{z=0} = I_R \, , \quad \text{(D-3)}$$

and

$$p_{II}(T, \infty) = 0 \, . \quad \text{(D-4)}$$

The signs in ( ) are to be used for $\sqrt{T_F} > z/2\sqrt{T_F}$ and $i \geq 1$. After taking the Laplace Transformation,

$$\frac{d^2P}{dz^2} - (1 + s)P = \frac{I_F}{2}\left(e^{-z} \sum_{i=0}^{N} (-)C_i e^{(-)a_i\left(\frac{z}{2\sqrt{T_F}} - \sqrt{T_F}\right)} - e^{z} \sum_{i=0}^{N} C_i e^{a_i\left(\frac{z}{2\sqrt{T_F}} + \sqrt{T_F}\right)}\right) , \quad \text{(D-5)}$$

$$- \frac{dP}{dz}\bigg|_{z = 0} = \frac{I_R}{s} , \tag{D-6}$$

and

$$P(s, \infty) = 0 , \tag{D-7}$$

where P is the transformed dependent variable.

The complete solution of this equation has the form of

$$P_{II}(s, z) = C_1 e^{z\sqrt{1 + s}} + C_2 e^{-z\sqrt{1 + s}}$$
$$+ \frac{I_F}{2} \left(\sum_{i = 0}^{N} (-) \frac{C_i \, e^{(\pm)a_i \sqrt{T_F}} \, e^{z\left(\frac{(-)a_i}{2\sqrt{T_F}} - 1\right)}}{\left(\frac{(-)a_i}{\sqrt{T_F}} - 1\right)^2 - (1 + s)}\right.$$
$$\left. - \sum_{i = 0}^{N} \frac{C_i \, e^{a_i\sqrt{T_F}} \, e^{z\left(\frac{a_i}{2\sqrt{T_F'}} + 1\right)}}{\left(\frac{a_i}{2\sqrt{T_F}} + 1\right)^2 - (1 + s)} \right) . \tag{D-8}$$

From the boundary conditions D-6 and D-7 the values of C_1 and C_2 may be found. The inverse transform of this equation may be found with the aid of pairs #825 and #819 of Reference 3. This results in

$$p_{II}(T,z) = -\frac{I_F}{2}\left\{\sum_{i=0}^{N}(-)C_i e^{(\pm)a_i\sqrt{T_F}}\; \frac{e^{\left[\left(\frac{(-)a_i}{2\sqrt{T_F}}-1\right)^2-1\right]T}}{2}\right.$$

$$\left[e^{-z\left(\frac{(-)a_i}{2\sqrt{T_F}}-1\right)} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \left(\frac{(-)a_i}{2\sqrt{T_F}}-1\right)\sqrt{T}\right) - e^{z\left(\frac{(-)a_i}{2\sqrt{T_F}}-1\right)}\right.$$

$$\left.\left(\operatorname{erfc}\left[\frac{z}{2\sqrt{T}} + \left(\frac{(-)a_i}{2\sqrt{T_F}}-1\right)\sqrt{T}\right] -2\right)\right] - \sum_{i=0}^{N}(-)C_i\, e^{a_i\sqrt{T_F}}$$

$$\frac{e^{\left[\left(\frac{a_i}{2\sqrt{T_F}}+1\right)^2-1\right]T}}{2}\left[e^{-z\left(\frac{a_i}{2\sqrt{T_F}}+1\right)} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \left(\frac{a_i}{2\sqrt{T_F}}+1\right)\sqrt{T}\right)\right.$$

$$\left.\left.- e^{z\left(\frac{a_i}{2\sqrt{T_F}}+1\right)} \left(\operatorname{erfc}\left[\frac{z}{2\sqrt{T}} + \left(\frac{a_i}{2\sqrt{T_F}}+1\right)\sqrt{T}\right] - 2\right)\right]\right\} - \frac{I_R}{2}\left[e^{-z}\right.$$

$$\left.\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z}\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right], \qquad \text{(D-9)}$$

where the signs in ( ) are to be used when $\sqrt{T_F} > z/2\sqrt{T_F}$ and $i \geq 1$.

To illustrate that this equation is in accordance with the steady state forward bias calculation, T_F will be considered very large in Equation D-9. This results in

$$p_{II}(T, z) = \frac{I_F}{2}\Big[\Big(1 - \sum_{i=1}^{N} C_i e^{a_i\sqrt{T_F}}\Big)\Big(\frac{e^z}{2}\left[\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right) - 2\right]$$

$$- \frac{e^{-z}}{2}\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right)\Big) + \Big(1 + \sum_{i=1}^{N} C_i e^{a_i\sqrt{T_F}}\Big)\Big(\frac{e^z}{2}\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)$$

$$- \frac{e^{-z}}{2}\left[\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - 2\right]\Big)\Big] + \frac{I_R}{2}\Big[e^{-z}\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right)$$

$$- e^{z}\operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\Big] \quad . \tag{D-10}$$

It may be seen that the above equation reduces exactly to that of the steady state forward bias case (Equation C-11).

E. Storage Period Following Finite Forward Bias Time (Using an Approximation)

The following equation is to be solved

$$\frac{\partial p_{II}}{\partial T} = \frac{\partial^2 p_{II}}{\partial z^2} - p_{II} \ . \tag{E-1}$$

With boundary conditions being

$$p_{II}(0, z) = m\ I_F\ e^{-rz} \ , \tag{E-2}$$

$$- \left.\frac{\partial p_{II}}{\partial z}\right|_{z=0} = I_R \ , \tag{E-3}$$

and

$$p_{II}(T, \infty) = 0 \ . \tag{E-4}$$

Taking the Laplace Transform

$$\frac{d^2P}{dz^2} - P\,(1 + s) = -\,m\,I_F\,e^{-rz}\,, \tag{E-5}$$

$$P(s, \infty) = 0, \tag{E-6}$$

and

$$-\left.\frac{dP}{dz}\right|_{z = 0} = \frac{I_R}{s}\,. \tag{E-7}$$

The solution to this differential equation is

$$P(s, z) = \frac{I_R\, e^{-z\sqrt{1 + s}}}{s\,\sqrt{1 + s}} - \frac{rm\, I_F\, e^{-z\sqrt{1 + s}}}{(s + 1 - r^2)\,\sqrt{1 + s}} + \frac{m\, I_F\, e^{-rz}}{(s + 1 - r^2)}\,. \tag{E-8}$$

The inverse transformation may be found with the aid of #825 and #438 of Reference 3. When these are performed, the above equation becomes:

$$p(T, z) = \frac{I_R}{2}\left[e^{-z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) - e^{z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right]$$

$$+ \frac{m\, I_F}{2}\, e^{-(1 - r^2)T}\left[e^{rz}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{T}} + r\sqrt{T}\right) - e^{-rz}\left(\mathrm{erfc}\left(\frac{z}{2\sqrt{T}} - r\sqrt{T}\right) - 2\right]\,. \tag{E-9}$$

F. Reverse Recovery Phase (Diffusion Current)

The solution to the following equation is to be obtained:

$$\frac{\partial p_{III}}{\partial T} = \frac{\partial^2 p_{III}}{\partial z^2} - p_{III}\,. \tag{F-1}$$

With the boundary conditions being

$$p(0, z) = \frac{I_F - I_R}{2}\left[e^{-z}\left(\sum_{i=0}^{N}(-)C_i e^{(-)a_i\left(\frac{z}{2\sqrt{T_s}} - \sqrt{T_s}\right)}\right)\right.$$

$$\left. - e^{z}\left(\sum_{i=0}^{N} C_i e^{a_i\left[\left(\frac{z}{2\sqrt{T_s}}\right)+\left(\sqrt{T_s}\right)\right]}\right)\right] + I_F e^{-z} , \qquad \text{(F-2)}$$

$$p(T, 0) = 0 , \qquad \text{(F-3)}$$

and

$$p(T, \infty) = 0 . \qquad \text{(F-4)}$$

Taking the Laplace Transform

$$\frac{d^2P}{dz^2} - P(1 + s) = \frac{I_R - I_F}{2}\left[e^{-z}\left(\sum_{i=0}^{N}(-)C_i e^{(-)a_i\left(\frac{z}{2\sqrt{T_s}} - \sqrt{T_s}\right)}\right)\right.$$

$$\left. - e^{z}\left(\sum_{i=0}^{N} C_i\, e^{a_i\left(\frac{z}{2\sqrt{T_s}} + \sqrt{T_s}\right)}\right)\right] - I_F e^{-z} , \qquad \text{(F-5)}$$

$$P(s, 0) = 0 , \qquad \text{(F-6)}$$

and

$$P(s, \infty) = 0 . \qquad \text{(F-7)}$$

The complete solution to this equation has the form

$$P(s, z) = C_1 e^{z\sqrt{1+s}} + C_2\, e^{-z\sqrt{1+s}} + \frac{I_R - I_F}{2}$$

$$\left(\sum_{i=0}^{N} \frac{(-)C_i\, e^{(\pm)a_i\sqrt{T_s}}\; e^{z\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)}}{\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)^2 - (1+s)} - \sum_{i=0}^{N} \frac{C_i e^{a_i\sqrt{T_s}}\; e^{z\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)}}{\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)^2 - (1+s)} \right) + I_F\,\frac{e^{-z}}{s} \qquad \text{(F-8)}$$

The undetermined constants C_1 and C_2 in Equation F-8 may be found by using boundary conditions F-6 and F-7. When this is done the inverse transformation may be found with the aid of pair numbers 819 and 438 of Reference 3. The following equation is the result.

$$p(T,z) = \frac{I_R - I_F}{2}\Bigg[\sum_{i=0}^{N} (-)C_i e^{(\pm)a_i\sqrt{T_s}}\, \frac{e^{\left[\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)^2 - 1\right] T}}{2} \Bigg(e^{-z\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)}$$

$$\operatorname{erfc}\left[\frac{z}{2\sqrt{T}} - \left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)\sqrt{T}\right] + e^{z\left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)} \left(\operatorname{erfc}\left[\frac{z}{2\sqrt{T}} + \left(\frac{(-)a_i}{2\sqrt{T_s}} - 1\right)\sqrt{T}\right] - 2\right)\Bigg)$$

$$- \sum_{i=0}^{N} C_i e^{a_i\sqrt{T_s}}\, \frac{e^{\left[\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)^2 - 1\right] T}}{2} \Bigg(e^{-z\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)} \operatorname{erfc}\Big[\frac{z}{2\sqrt{T}}$$

$$- \left(\frac{a_i}{2\sqrt{T_s}} + 1\right)\sqrt{T}\Big] + e^{z\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)} \left(\operatorname{erfc}\left[\frac{z}{2\sqrt{T}} + \left(\frac{a_i}{2\sqrt{T_s}} + 1\right)\sqrt{T}\right] - 2\right)\Bigg)\Bigg] - \frac{I_F}{2}$$

$$\left[e^{-z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} - \sqrt{T}\right) + e^{z} \operatorname{erfc}\left(\frac{z}{2\sqrt{T}} + \sqrt{T}\right)\right] + I_F\, e^{-z} \quad . \qquad \text{(F-9)}$$

The signs in ( ) are to be used if $\sqrt{T_s} > z/2\sqrt{T_s}$ and $i \geq 1$.

Because of the numerous exponentials which may be raised to powers which contain positive coefficients times the variable, it seems beneficial to briefly illustrate the boundness of this equation. The terms of concern may be represented in the following form:

$$A_i \frac{I_R - I_F}{2} \left(e^{B_i T} e^{C_i z} \left[2 - \text{erfc} \left(\frac{z}{2\sqrt{T}} + C_i \sqrt{T} \right) \right] \right) \tag{F-10}$$

where

$$B_i = \left(\frac{a_i}{2\sqrt{T_s}} - 1 \right)^2 - 1 \quad \text{and} \quad C_i = \left(\frac{a_i}{2\sqrt{T_s}} - 1 \right) .$$

If the B_i's are negative, it may be shown that Equation F-10 is well bounded for all z. This may be done with the aid of the asymptotic approximations given in Equation 48 and Equation 49 of this thesis.

If the B_i's are positive and the C_i's are negative one may, for very large values of T, use the asymptotic approximation and write Equation F-10 as

$$\frac{I_R - I_F}{2} A_i \frac{e^{B_i T} e^{-C_i^2 T}}{|C_i| \sqrt{\pi T}} . \tag{F-11}$$

Now from the definitions of B_i and C_i, this equation becomes

$$\frac{I_R - I_F}{2} A_i \frac{e^{-T}}{|C_i| \sqrt{T\pi}} , \tag{F-12}$$

which approaches zero for large values of T.

1. <u>Current during the recovery phase</u>

Of prime concern during Phase III is the diffusion current. The magnitude of this current may be calculated by the following equation:

$$I = - \left. \frac{\partial p}{\partial z} \right|_{z = 0} . \tag{F-13}$$

Hence, the slope of Equation F-9 is to be found at the junction. In taking this derivative, the following equation will be used:

$$d(\text{erf } x) = \frac{2\, e^{-x^2}}{\sqrt{\pi}} \quad dx\ . \tag{F-14}$$

Following this procedure, the following equation was obtained

$$I(T) = \frac{I_R - I_F}{2}\Bigg[\sum_{i=0}^{N} (-)C_i e^{(\pm)a_i\sqrt{T_s}}\ \frac{e^{[(\frac{(-)a_i}{2\sqrt{T_s}} - 1)^2 - 1]\,T}}{2}\ \Bigg(\Big[\frac{(-)a_i}{2\sqrt{T_s}} - 1\Big]$$

$$\Big(2 - \text{erfc}\ \big(\frac{(-)a_i}{2\sqrt{T_s}} - 1\big)\sqrt{T}\Big) + \big(\frac{(-)a_i}{2\sqrt{T_s}} - 1\big)\ \text{erfc}\Big[-\big(\frac{(-)a_i}{2\sqrt{T_s}} - 1\big)\sqrt{T}\Big]$$

$$+ \frac{2\, e^{-(\frac{(-)a_i}{2\sqrt{T_s}} - 1)^2\, T}}{\sqrt{\pi T}}\Bigg) + \sum_{i=0}^{N} C_i e^{a_i\sqrt{T_s}}\ \frac{e^{[(\frac{a_i}{2\sqrt{T_s}} + 1)^2 - 1]\,T}}{2}$$

$$\Bigg(\Big[\frac{a_i}{2\sqrt{T_s}} + 1\Big]\Big(2 - \text{erfc}\ \big(\frac{a_i}{2\sqrt{T_s}} + 1\big)\sqrt{T}\Big) + \big(\frac{a_i}{2\sqrt{T_s}} + 1\big)\ \text{erfc}\Big[-\big(\frac{a_i}{2\sqrt{T_s}} + 1\big)\sqrt{T}\Big]$$

$$+ \frac{2\, e^{-(\frac{a_i}{2\sqrt{T_s}} + 1)^2\ T}}{\sqrt{\pi T}}\Bigg)\Bigg] + \frac{I_F}{2}\Big(\text{erfc}\ \sqrt{T} - \text{erfc}(-\sqrt{T}) - \frac{2e^{-T}}{\sqrt{\pi T}}\Big) + I_F\ . \tag{F-15}$$

For the special case of $T_s = 0$, the signs in ( ) are not used and the above equation reduces to

$$I(T) = -\ I_F\ \Big(\frac{e^{-T}}{\sqrt{\pi T}} + \text{erf}\ \sqrt{T} - 1\Big)\ . \tag{F-16}$$

For the general case of $T_s > 0$, the signs in ( ) are used. This results in

$$I(T) = (I_F - I_R) \sum_{i=1}^{N} C_i e^{a_i \sqrt{T_s}} e^{\left[\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)^2 - 1\right] T}$$

$$\left(\frac{e^{-\left(\frac{a_i}{2\sqrt{T_s}} + 1\right)^2 T}}{\sqrt{\pi T}} + \left(\frac{a_i}{2\sqrt{T_s}} + 1\right) \operatorname{erf}\left(\left(\frac{a_i}{2\sqrt{T_s}} + 1\right) \sqrt{T}\right)\right)$$

$$+ I_R - I_F \left(\frac{e^{-T}}{\sqrt{\pi T}} + \operatorname{erf} \sqrt{T}\right) . \qquad \text{(F-17)}$$

G. Potential Gradient in a Finite Length N-Type Region

1. Forward bias

The solution to the following time independent equation is desired:

$$0 = \frac{d^2 p_I}{dz^2} - fL_p \frac{dp_I}{dz} - p_I . \qquad \text{(G-1)}$$

The boundary conditions being

$$\frac{dp}{dz} - fL_p\, p = - I_F \text{ at } z = 0 , \qquad \text{(G-2)}$$

and

$$\frac{dp}{dz} + \left(\frac{L_p S}{D_p} - fL_p\right)p = 0 \text{ at } z = W . \qquad \text{(G-3)}$$

The solution to this equation has the following form:

$$p(z) = C_1\, e^{\left(\frac{fL_p}{2} + \sqrt{1 + \frac{f^2 L_p^2}{4}}\right)z} + C_2\, e^{\left(\frac{fL_p}{2} - \sqrt{1 + \frac{f^2 L_p^2}{4}}\right)z} \qquad \text{(G-4)}$$

To simplify the writing of the equations, the quantities A, B, and C will be defined as follows:

$$A = \frac{fL_p}{2} \qquad \text{(G-5)}$$

$$B = \sqrt{1 + \frac{f^2 L_p^{\,2}}{4}} \quad , \tag{G-6}$$

and

$$C = \frac{SL_p}{D_p} \quad . \tag{G-7}$$

The boundary conditions (Equations G-2 and G-3) of the following two equations may be solved for the values of C_1 and C_2.

$$(A - B)\ C_1 + (A + B)\ C_2 = I_F \tag{G-8}$$

$$(-A + B + C)\ e^{(A + B)W} C_1 - (A + B - C)\ e^{(A - B)W}\ C_2 = 0 \tag{G-9}$$

Upon solution of these equations and substitution into Equation G-4, one obtains:

$$p(z) = I_F\ \frac{(A + B - C)\ e^{Az + B(z - 2W)} + (-A + B + C)\ e^{(A - B)z}}{(A + B)(-A + B + C) + (A - B)(A + B - C)\ e^{-2BW}} \tag{G-10}$$

When the case of large W is considered, Equation G-10 becomes:

$$p(z) = I_F\ \frac{e^{(A - B)z}}{(A + B)} \quad . \tag{G-11}$$

For the case of no drift field (f = 0), Equation G-10 reduces to:

$$p(z) = +\ I_F\ \frac{(1 + C\ e^{-z} + (1 - C)\ e^{z-2W}}{1 + C - (1 - C)\ e^{-2W}} \quad . \tag{G-12}$$

2. Reverse bias

a) Finite length n-type region with no drift field For this case, the solution to the following equation is desired:

$$\frac{\partial p_{II}}{\partial T} = \frac{\partial^2 p_{II}}{\partial z^2} - p_{II} \ . \tag{G-13}$$

With the boundary conditions being

$$p(0,\ z) = I_F \frac{(1 + C)\ e^{-z} + (1 - C)\ e^{z-2W}}{1 + C - (1 - C)\ e^{-2W}} \tag{G-14}$$

(for which the quantities M and N will be defined such that the following is true):

$$p(0,\ z) = I_F\ (M\ e^{-z} + N\ e^{z})\ , \tag{G-15}$$

$$- \frac{\partial p_{II}}{\partial z} = I_R \text{ at } z = 0\ , \tag{G-16}$$

and

$$\frac{\partial p_{II}}{\partial z} + \frac{SL_p}{D_p}\ p_{II} = 0 \text{ at } z = W\ . \tag{G-17}$$

After taking the Laplace Transform,

$$\frac{d^2P}{dz^2} - P(1 + s) = -I_F(M\ e^{-z} + N\ e^{z}), \tag{G-18}$$

$$- \frac{dP}{dz} = \frac{I_R}{s} \text{ at } z = 0\ , \tag{G-19}$$

and

$$\frac{dP}{dz} + \frac{SL_p}{D_p} P = 0 \text{ at } z = W\ . \tag{G-20}$$

The solution to this equation has the following form:

$$P(s,\ z) = C_1\ e^{z\sqrt{1 + s}} + C_2\ e^{-z\sqrt{1 + s}} + \frac{I_F}{s}\ (M\ e^{-z} + N\ e^{z}) \tag{G-21}$$

From the boundary conditions (Equations G-19 and G-20) the values of C_1 and C_2 may be found. When these are substituted into Equation G-21, the result is

$$P(s,z) = \frac{[I_R + I_F(N-M)]\;[(\sqrt{1+s} - \frac{SL_p}{D_p})e^{(z-2W)\sqrt{1+s}} + (\frac{SL_p}{D_p} + \sqrt{1+s})e^{-z\sqrt{1+s}})]}{s\sqrt{1+s}\;[\frac{SL_p}{D_p} + \sqrt{1+s} + (\frac{SL_p}{D_p} - \sqrt{1+s})\; e^{-2W\sqrt{1+s}}]}$$

$$+ \frac{I_F}{s}(M\,e^{-z} + N\,e^{z}) \; . \qquad \text{(G-22)}$$

In this section, the storage time is of interest. Hence, an equation of $p(T_s, 0)$, which is relatively easy to solve for T_s, is desired. To do this, the inverse of the above equation will be made easier by employing the following approximations:

$$1 - e^{-2W\sqrt{1+s}} \approx 1, \qquad \text{(G-23)}$$

and

$$1 + e^{-2W\sqrt{1+s}} \approx 1. \qquad \text{(G-24)}$$

These approximations will not seriously limit the application of this material since Equations G-23 and G-24 are true for most commercially available diodes.

Using these conditions, the inverse of Equation G-22 becomes

$$p(T,z) = \frac{I_R + I_F(N-M)}{2}\left[e^{-z}\;\text{erfc}\;(\frac{z}{2\sqrt{T}} - \sqrt{T}) - e^{z}\;\text{erfc}\;(\frac{z}{2\sqrt{T}} + \sqrt{T})\right]$$

$$+ I_F\,(M\,e^{-z} + N\,e^{z}) \; . \qquad \text{(G-25)}$$

The storage time may now be found by letting $p(T_s, 0) = 0$. Upon solving and substituting in the values of M and N, one obtains:

$$\operatorname{erf}\sqrt{T_s} = \frac{\dfrac{1 + SL_p/D_p + (1 - SL_p/D_p)\, e^{-2W}}{1 + SL_p/D_p - (1 - SL_p/D_p)\, e^{-2W}}}{1 + \theta} \tag{G-26}$$

b) Large W with a drift field For this case, the following equation will be solved:

$$\frac{\partial p}{\partial T} = \frac{\partial^2 p}{\partial z^2} - fL_p \frac{\partial p}{\partial z} - p \, . \tag{G-27}$$

With the boundary conditions being

$$p(0, z) = \frac{I_F}{B + A} \quad e^{z(A - B)} \, , \tag{G-28}$$

$$p(T, \infty) = 0 \, , \tag{G-29}$$

and

$$- \frac{\partial p}{\partial z} + (fL_p)\, p = I_R \text{ at } z = 0. \tag{G-30}$$

Taking the Laplace Transform:

$$\frac{d^2P}{dz^2} - fL_p \frac{dP}{dz} - P(1 + s) = I_F \frac{e^{z(A - B)}}{A + B} \, , \tag{G-31}$$

$$P(s, \infty) = 0, \tag{G-32}$$

and

$$- \frac{dP}{dz} + fL_p P = \frac{I_R}{s} \text{ at } z = 0 \, . \tag{G-33}$$

The solution to this equation has the following form:

$$P(s,z) = C_1 \, e^{z(A + \sqrt{B^2 + s})} + C_2 \, e^{(A - \sqrt{B^2 + s})} + I_F \frac{e^{z(A-B)}}{s(A+B)} \, . \tag{G-34}$$

The values of C_1 and C_2 may be found by making use of the boundary conditions (Equations G-32 and G-33). When these are substituted into Equation G-34 the inverse may be found with the aid of part #825 in Reference 3. This results in

$$p(T, z) = \left[I_R - I_F\right] e^{Az} \left[\frac{e^{-Bz}}{2(A + B)} \operatorname{erfc} \left(\frac{z}{2\sqrt{T}} - B\sqrt{T}\right) + \frac{e^{zB}}{2(A - B)} \operatorname{erfc} \left(\frac{z}{2\sqrt{T}} + B\sqrt{T}\right) + Ae^{-T} \operatorname{erfc} \left(\frac{z}{2\sqrt{T}} + A\sqrt{T}\right)\right] + I_F \frac{e^{z(A - B)}}{(B + A)} \tag{G-35}$$

The equation to determine the storage time is determined by using the condition that $p(T_s, 0) = 0$. When this is applied to Equation G-35 the following is the result:

$$\frac{1}{1 + \theta} = \left[\frac{fL_p}{2} + \sqrt{1 + \frac{f^2L_p^2}{4}}\right]\left[\frac{fL_p}{2} (e^{-T_s} - 1) + \sqrt{1 + \frac{f^2L_p^2}{4}} \operatorname{erf} \sqrt{\left(1 + \frac{f^2L_p^2}{4}\right)T_s} - \frac{fL_p}{2} e^{-T_s} \operatorname{erf} \frac{fL_p}{2} \sqrt{T_s}\right] . \tag{G-36}$$